BOSON INTERFEROMETRY IN HIGH-ENERGY PHYSICS

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Abstract

Intensity interferometry and in particular that based on Bose–Einstein correlations (BEC) constitutes at present the only direct experimental method for the determination of sizes and lifetimes of sources in particle

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and nuclear physics. The measurement of these is essential for an understanding of the dynamics of strong interactions which are responsible for the existence and properties of atomic nuclei. Moreover, a new state of matter, quark matter, in which the ultimate constituents of matter move freely, is within the reach of present accelerators or those under construction. The confirmation of the existence of this new state is intimately linked with the determination of its space–time properties. Furthermore, BEC provides information about quantum coherence which lies at the basis of the phenomenon of Bose–Einstein condensation seen in many chapters of physics. Coherence and the associated classical fields are essential ingredients in modern theories of particle physics including the standard model. Last but not least besides this "applicative" aspect of BEC, this effect has implications for the foundations of quantum mechanics including the understanding of the concept of "identical particles". Recent theoretical developments in BEC are reviewed and their application in high-energy particle and heavy-ion reactions is analysed. The treated topics include: (a) a comparison between the wave-function approach and the space–time approach based on classical currents, which predicts "surprising" particle–anti-particle BEC, (b) the study of final state interactions, (c) the use of hydrodynamics, and (d) the relation between correlations and multiplicity distributions.

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1. Introduction

The method of photon intensity interferometry was invented in the mid-1950s by Hanbury–Brown and Twiss for the measurement of stellar dimensions and is sometimes called the HBT method. In 1959–1960 G. Goldhaber, S. Goldhaber, W. Lee and A. Pais discovered that identical charged pions produced in $\bar{p}$–p annihilation are correlated (the GGLP effect). Both the HBT and the GGLP effects are based on Bose–Einstein correlations (BEC). Subsequently Fermi–Dirac correlations for nucleons have also been observed. Loosely speaking, both these correlation effects can be viewed as a consequence of the symmetry (antisymmetry) properties of the wave function with respect to permutation of two identical particles with integer (half-integer) spin and are thus intrinsic quantum phenomena. At a higher level, these symmetry properties of identical particles are expressed by the commutation relations of the creation and annihilation operators of particles in the second quantisation (quantum field theory). The quantum field approach is the more general approach as it contains the possibility to deal with creation and annihilation of particles and certain phenomena like the correlation between particles and antiparticles can be properly described only within this formalism. Furthermore, at high energies, because of the large number of particles produced, not all particles can be detected in a given reaction and therefore one measures usually only inclusive cross sections. For these reactions the wave-function formalism is impractical. Related to this is the fact that the second quantisation provides through the density matrix a transparent link between correlations and multiplicity distributions. This last topic has been in the centre of interest of multiparticle dynamics for the last 20 years (we refer among other things to Koba–Nielsen–Olesen (KNO) scaling and "intermittency"). Furthermore, one of the most important properties of systems made of identical bosons which is responsible for the phenomenon of *lasing* is quantum statistical coherence. This feature is also not accessible to a theoretical treatment except in field theory.

The present review is restricted to Bose–Einstein correlations which constitute by far the majority of correlations papers both of theoretical and experimental nature. This is due to the fact that BEC present important heuristic and methodological advantages over Fermi–Dirac correlations. Among the first we mention the fact that quantum coherence appears only in BEC. Among the second, one should recall that pions are the most abundantly produced secondaries in high-energy reactions.

In the last few years there has been a considerable surge of interest in boson interferometry. This can be judged by the fact that at present there is no meeting on multiparticle production where numerous contributions to this subject are presented. Moreover, since 1990 [1] meetings dedicated exclusively to this topic are organised; this reflects the realisation that BEC on their own are an important subject. This development is in part due to the fact that at present intensity interferometry constitutes the only direct experimental method for the determination of sizes and lifetimes of sources in particle and nuclear physics. Since soft strong interactions which are responsible for multiparticle production processes cannot be treated by perturbative QCD, phenomenological approaches have to be used in this domain and space–time concepts are essential elements in these approaches. That is why intensity interferometry has become an indispensable tool in the investigation of the dynamics of high-energy reactions.

However, this alone could not explain the explosion of interest in BEC if one did not take into account the search for quark–gluon plasma which has mobilised the attention of most of the nuclear physics and of an appreciable part of the particle physics community. For several reasons the space–time properties of "fireballs" produced in heavy ion reactions are key to the process of understanding whether quark matter has been formed.

The main emphasis of the present review will be on theoretical developments which took place after 1989–1990. Experimental results will be mentioned only insofar as they illustrate the theoretical concepts. For a review of older references see. e.g. the paper by Boal et al. [2]. In the 1990s the single most important theoretical event was in our view the space–time generalisation of the classical current formalism. For a detailed presentation of this generalisation and its applications up to 1993 the reader is referred to Ref. [3]. For a review of experimental results in e^+–e^- reactions see [4]. Finally a more pedagogical and more complete treatment of the theory of BEC can be found in [5].

There are two categories of papers not mentioned: (i) Those which the reviewer was unaware of; he apologises to the authors of these papers for this. (ii) Those which he considered as irrelevant or as repetitions of previous work. The large number of papers quoted despite these restrictions shows that an exhaustive listing of references on BEC is not trivial.

2. The GGLP effect

In the period 1954–1960 Hanbury–Brown and Twiss developed the method of optical intensity interferometry for the determination of (angular sizes) of stars (see e.g. [6]). The particle physics equivalent of the Hanbury–Brown–Twiss (HBT) effect in optics is the Goldhaber, Goldhaber, Lee and Pais (GGLP) effect [7,8] which we shall describe schematically below.

However, when going over from optics to particle physics the following point has to be considered: in particle physics one does not measure distances r in order to deduce (differences of) momenta k and thus angular sizes, but one measures rather momenta in order to deduce distances. This explains why GGLP were not aware of the HBT method.[1]

Up to a certain point there are two ways of approaching the theory of intensity interferometry: the wave-function approach and the field theoretical approach. Although the first one is only a particular case of the second one it is still useful because it allows sometimes a more intuitive understanding of certain concepts and in particular that of the distinction between boson and fermion correlations. We shall start therefore with the wave-function approach.

2.1. The wave-function approach

Let us consider for the beginning a source which consists of a number of discrete emission points i each of which is characterised by a probability amplitude

$$F_i(\boldsymbol{r}) = F_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) \ . \tag{2.1}$$

[1] For a comparison of optical and particle physics intensity interferometry see e.g. [5].

Let $\psi_{\boldsymbol{k}}(\boldsymbol{r})$ be the wave function of an emitted particle. The total probability $P(\boldsymbol{k})$ to observe the emission of one particle with momentum $\boldsymbol{k}$ from this source is obtained by summing the contributions of all points i. If this summation is done *incoherently* the single particle probability reads

$$P(\boldsymbol{k}) = \sum_i |F_i \psi(r_i)|^2 \ . \tag{2.2}$$

Instead of discrete emission points consider now a source the emission points of which are continuously distributed in space and assume for simplicity that the wave functions are plane waves $\psi_{\boldsymbol{k}}(\boldsymbol{r}) \sim \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{r})$. The sum over x_i will now be replaced by an integral and we have

$$P(\boldsymbol{k}) = \int |F(\boldsymbol{r})|^2 \, \mathrm{d}^3 r \ . \tag{2.3}$$

Similarly, the probability to observe two particles with momenta $\boldsymbol{k}_1, \boldsymbol{k}_2$ is

$$P(\boldsymbol{k}_1 \boldsymbol{k}_2) = \int |\psi_{1,2}|^2 f(\boldsymbol{r}_1) f(\boldsymbol{r}_2) \, \mathrm{d}^3 r_1 \, \mathrm{d}^3 r_2 \ . \tag{2.4}$$

Here $\psi_{1,2} = \psi_{1,2}(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{r}_1, \boldsymbol{r}_2)$ is the two-particle wave function and we have introduced the density distribution $f = |F|^2$.

Suppose now that the two particles are identical. Then the two-particle wave function has to be symmetrised or antisymmetrised depending on whether we deal with identical bosons or fermions. Assuming plane waves we have

$$\psi_{1,2} = (1/\sqrt{2})[\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1 \boldsymbol{r}_1 + \boldsymbol{k}_2 \boldsymbol{r}_2)} \pm \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1 \boldsymbol{r}_2 + \boldsymbol{k}_2 \boldsymbol{r}_1)}] \tag{2.5}$$

with the plus sign for bosons and the minus sign for fermions. With this wave function one obtains

$$P(\boldsymbol{k}_1, \boldsymbol{k}_2) = |\tilde{f}_I(0)|^2 \pm |\tilde{f}_I(\boldsymbol{k}_1 - \boldsymbol{k}_2)|^2 \ . \tag{2.6}$$

The incoherent summation corresponds to random fluctuations of the amplitudes F_i.

Following the GGLP experiment [8] consider the space points x_1 and x_2 within a source, so that each point emits two identical particles (equally charged pions in the case of GGLP) with momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. These particles are detected in the registration points r_1 and r_2 so that in r_1 only particles of momentum k_1 and in r_2 only particles of momentum k_2 are registered (see Fig. 1). Because of the identity of particles one cannot decide which particle pair originates in x_1 and which in x_2.

Assuming that the individual emission points of the source act incoherently GGLP derived Eq. (2.6) which for bosons leads to the second-order correlation function

$$C_2(\boldsymbol{k}_1, \boldsymbol{k}_2) \equiv P_2(\boldsymbol{k}_1, \boldsymbol{k}_2)/P_1(\boldsymbol{k}_1) P_1(\boldsymbol{k}_2) = 1 + |\tilde{f}(\boldsymbol{q})|^2 \ , \tag{2.7}$$

where $\tilde{f}$ is the Fourier transform of f and $\boldsymbol{q}$ the momentum difference $\boldsymbol{k}_1 - \boldsymbol{k}_2$.

Eq. (2.7) shows quite clearly how in particle physics momentum (correlation) measurements can yield information about the space–time structure of the source.

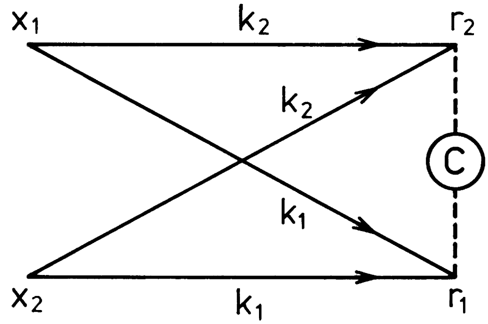

Fig. 1. The GGLP experiment schematically.

In the case of coherent summation one gets instead $C_2 = 1$ (see e.g. [2]). We deduce from this result (see also below) that coherence reduces the correlation and that a purely coherent source has a correlation function which does not depend on its geometry.[2]

2.1.1. Newer correlation measurements in $\bar{p}$–p annihilation

The original GGLP experiment [7] measured the correlation function in terms of the opening angle[3] of a pion pair.

The GGLP experiment has been repeated in the last few years at LEAR (see e.g. [12] where also older references are quoted[4]). In these newer experiments the three momenta of particles were measured, although one continued to use as a variable for the correlation function the invariant momentum difference Q, as suggested already in [8].

One of the remarkable observations made in all annihilation reactions is that the intercept of the second-order correlation function $C_2(k, k)$ appears consistently to exceed the canonical value of 2 reaching values up to 4. (This effect was possibly not seen in the original experiment [7] because of the averaging over the magnitudes of the momenta.)

In [14] this effect was attributed to resonances while subsequently [15,12] a (non-chaotic) Skyrmion-type superposition of coherent states was proposed as an alternative explanation.

Another possible explanation of this intriguing observation may be that in annihilation processes squeezed states (see Section 2.2) are produced, while this is not the case in other processes. Indeed, it is known that squeezed states can lead to overbunching effects (see below). Furthermore, as shown in Ref. [16], squeezed states may be preferentially produced in rapid reactions. Or, according to some authors [15], annihilation is a more rapid process than other reactions

[2] See also [9,10] for an attempt to approach the issue of coherence and chaos by using wave packets.

[3] The use of the opening angle as a kinematical variable in BEC studies was readopted in Ref. [11] where it was recommended as a tool for the investigation of final state interactions.

[4] For a reanalysis of the results of an older annihilation experiment see [13].

occurring at higher energies. Given the importance of squeezed states, further experimental and theoretical studies of this issue are highly desirable.[5]

2.1.2. *Resonances, apparent coherence and other experimental problems: the λ factor*

It is known that in multiparticle production processes, an appreciable fraction of pions originates from resonance decays. Resonances act in opposite ways on the correlation function of pion pairs. On the one hand, the interference between pions originating from short-lived resonances and "directly" produced pions leads to a narrow peak in the second-order correlation function at small values of q [17,18]. On the other hand, long-lived resonances give rise to pions which are beyond the range of detectors and this leads to an apparent decrease of the intercept of the second-order correlation function $C_2(k,k)$. These modifications of the intercept are important among other things, because their understanding is essential in the search for coherence through the intercept criterion. As mentioned already one of the most immediate consequences of coherence is the decrease of the correlation function at small q.

Because of the large number of different resonances produced in high-energy reactions a quantitative estimate of their effects is possible only via numerical techniques. In Sections 4.9.1 and 5.1.3 we present a more detailed discussion of the influence of resonances on BEC within the Wigner function formalism. For older references on this topic see also [2,19,20].

Another experimental difficulty is that, because of limited statistics or certain technical problems, sometimes not all degrees of freedom can be measured in a given experiment. This leads to an effective averaging over the non-measured degrees of freedom and hence also to an effective reduction of the correlation function. As a matter of fact BEC experiments have shown from the very beginning that the extrapolation of the correlation function to $q=0$ usually never led to the maximum value $C_2(0)=2$ permitted by Eq. (2.7).

To take into account empirically this effect experimentalists introduced into the correlation function a correction factor λ. Thus Eq. (2.7), e.g. was modified into

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \lambda|\tilde{f}(\boldsymbol{q})|^2 \ . \tag{2.8}$$

Because one initially thought that formally this generalisation offers also the possibility to describe partially coherent sources, the corresponding parameter λ was postulated to be limited by (0,1) and called the "incoherence" factor; indeed $\lambda=0$ leads to a totally coherent source and $\lambda=1$ to a totally chaotic one. Unfortunately, this nomenclature is not quite correct, as explained below. It should also be mentioned that there exists a strong correlation between the empirical values of λ and those of the "radius" which enter $\tilde{f}$ (see e.g. [21] for a special study of this issue).

2.1.3. *The limitations of the wave-function formalism*

The wave-function formalism presented in the previous subsection has severe shortcomings:

(a) The correlation function (2.7) depends only on the difference of momenta q and not also on the sum $\boldsymbol{k}_1+\boldsymbol{k}_2$, in contradiction with experimental data. Although we will see in Section 4.3 that this limitation disappears automatically in the quantum statistical (field theoretical) approach, it

[5] In Ref. [16] the effect of chaotic superpositions of squeezed states in BEC was also studied. It is shown that such a superposition always leads to *overbunching*, while pure squeezed states can lead also to antibunching.

can also be remedied within the first quantisation formalism by using the Wigner function approach. In this approach the source function is from the beginning a function both of coordinates and momenta and therefore the correlation function depends on k_1 and k_2 separately. There is, of course, a price to pay for this procedure as it involves a semiclassical approximation.[6]

(b) The wave-function formalism may be useful when exclusive reactions are considered as was the case, e.g. in the GGLP paper. Indeed $\psi_{1,2}$ in Eq. (2.5) is just the wave function of the two-boson system, i.e. an assumption that two and only two bosons are produced is made. At low energies, i.e. low average multiplicities, this condition can be satisfied. However at high energies the identification of all particles is very difficult and up to now has not been done. Therefore, one measures practically always inclusive cross sections. This means that instead of (2.5) one would have to use a wave function which describes the two bosons in the presence of all other produced particles. To obtain such a wave function one would have to solve the Schrödinger equation of the many body, strongly interacting system, which is not a very practical proposal. Related to this is the difficulty to treat higher-order correlations within the wave-function formalism.

(c) The correlation function C_2 in the wave-function formalism is independent of isospin and is thus the same for charged and neutral particles. We shall see in Section 4.4 that in a more correct quantum field theoretical approach, this is not the case. This will affect among other things the bounds of the correlations and will lead to quantum statistical particle–antiparticle correlations which are not expected in the wave-function formalism.

(d) Coherence cannot be treated adequately (see below). The correlation functions derived in this subsection refer in general to incoherent sources and attempts to introduce coherence within the wave-function formalism are rather ad hoc parametrisations. However, coherence is the most characteristic and important property of Bose–Einstein correlations among other things because it is the basis of the phenomenon of Bose–Einstein *condensates* found in many chapters of physics, like superconductors, superfluids, lasers, and the recently discovered atomic condensates [24]. It would be very surprising if coherence would not be found also in particle physics given the fact that the wavelengths of the emitted particles are of the same order as that of the sources. Furthermore, as pointed out in this connection in [25] modern particle physics is based on spontaneously broken symmetries. The associated fields are coherent. That is why one of the main motivations of BEC research should be the measurement of the amount of coherence in strong interactions. For this purpose the formalism of BEC has to be generalised to include the presence of (partial) coherence and this again can be done correctly only within quantum statistics, i.e. quantum field theory.

We conclude this subsection with the observation that the wave function or wave packet approach may, nevertheless, be useful in BEC for the investigation of final state interactions (see below) or for the construction of event generators, where phases or quantum amplitudes are ignored anyway. Also, for correlations between fermions where coherence is absent the wave-function formalism may be an adequate substitute, although here too a field theoretical approach is possible.

[6] String models [22,23] also use a Wigner-type formalism. Here, it is postulated that there exists a "formation" time τ and therefore the particle production points are distributed around $t^2 - x^2 = \tau^2$. This implies among other things a correlation between particle production points and momenta.

2.2. Quantum optical methods in BEC

In high-energy processes in which the pion multiplicity is large, we may in general expect the methods of quantum statistics (QS)[7] to be useful. For BEC in particular they turn out to be indispensable. These methods have been applied with great success particularly in quantum optics (QO), superfluidity, superconductivity, etc. What distinguishes optical phenomena from those in particle physics are conservation laws and final state interactions which are present in hadron physics. At high energies and high multiplicities the first are not important. Neglecting for the moment the final state interactions also, QS reduces then to QO and we may take over the formalism of QO to interpret the data on multipion production at high energies, provided we consider identical pions. Given the general validity of QS (or QO), it is then clear that any model of multiparticle production must satisfy the laws of quantum statistics and this has far reaching consequences, independent of the particular dynamical mechanism which governs the production process. The main tools in the QO formalism are defined below.[8]

Coherent states and squeezed coherent states: Coherent states $|\alpha\rangle$ are eigenstates of the (one-particle) annihilation operator a

$$a|\alpha\rangle = \alpha|\alpha\rangle \ . \tag{2.9}$$

Squeezed coherent states are eigenstates $|\beta\rangle_s$ of the two-particle annihilation operator

$$b = \mu a + \nu a^\dagger \tag{2.10}$$

with

$$|\mu|^2 - |\nu|^2 = 1 \ , \tag{2.11}$$

so that

$$b|\beta\rangle_s = \beta|\beta\rangle_s \ . \tag{2.12}$$

One of the remarkable properties of these states which explains also their name is that for them the uncertainty in one variable can be *squeezed* at the expense of the other so that

$$(\Delta q)_s^2 \le 1/2\omega, \qquad (\Delta p)_s^2 \ge \omega/2 \tag{2.13}$$

or vice versa. The importance of this remarkable property lies among other things in the possibility to reduce quantum fluctuations and this explains the great expectations associated with them in communication and measurement technology as well as their interest from a heuristic point of view.

It has been found recently in [16] that squeezed states appear naturally when one deals with rapid phase transitions (explosions). Indeed consider the transition from a system a to a system b and assume that it proceeds rapidly enough for the relation between the creation and annihilation

[7] By QS we understand in the following the density matrix formalism within second quantisation.

[8] A review of the applications of quantum optical methods to multiparticle production up to 1988 can be found in [26].

operators and the corresponding fields in the two "phases" to remain unchanged. Mathematically, this process will be described by postulating at the moment of this transition the following relations between the generalised coordinate q and the generalised momentum p of the field:

$$\begin{aligned} q &= (1/\sqrt{2E_b})(b^\dagger + b) = (1/\sqrt{2E_a})(a^\dagger + a) , \\ p &= \mathrm{i}\sqrt{(E_b/2)}(b^\dagger - b) = \mathrm{i}\sqrt{(E_a/2)}(a^\dagger - a) , \end{aligned} \tag{2.14}$$

$a^\dagger, a$ are the free field creation and annihilation operators in the "phase a" and $b^\dagger, b$ the corresponding operators in the "phase b". Eq. (2.14) holds for each mode p. Then we get immediately a connection between the a and b operators,

$$\begin{aligned} a &= b\cosh r + b^\dagger \sinh r , \\ a^\dagger &= b\sinh r + b^\dagger \cosh r \end{aligned} \tag{2.15}$$

with

$$r = r(p) = \tfrac{1}{2}\log(E_a/E_b) . \tag{2.16}$$

Transformation (2.15) is just the squeezing transformation (2.10) with

$$\mu = \cosh r, \qquad \nu = \sinh r \tag{2.17}$$

which proves the statement made above.

The observation of squeezed states in BEC may thus serve as a signal for such rapid transitions. Furthermore, the existence of isospin induces in hadronic BEC certain effects which are specific for squeezed states. This topic will be discussed in Section 4.4.

From the point of view of BEC what distinguishes ordinary coherent states from squeezed states is the following: for coherent states the intercept $C_2(k,k) = 1$ while for squeezed states it can take arbitrary values. In Fig. 2 one can see such an example.

Expansions in terms of coherent states. Coherent states form an (over)complete set so that an arbitrary state $|f\rangle$ can be expanded in a unique way in terms of these states.

Of particular use is the expansion of the density matrix ρ in terms of coherent states. For a pure coherent state the density operator reads

$$\rho = |\alpha\rangle\langle\alpha| . \tag{2.18}$$

For an arbitrary density matrix case we have

$$\rho = \int \mathscr{P}(\alpha)|\alpha\rangle\langle\alpha|\, \mathrm{d}^2\alpha . \tag{2.19}$$

Here $\mathscr{P}$ is a weight function which usually, but not always, has the meaning of a probability. The normalisation condition for the density operator translates in terms of the $\mathscr{P}$ representation as

$$tr\,\rho = \int \mathscr{P}(\alpha)\, \mathrm{d}^2\alpha = 1 . \tag{2.20}$$

Eq. (2.19) is called also the Glauber–Sudarshan representation.

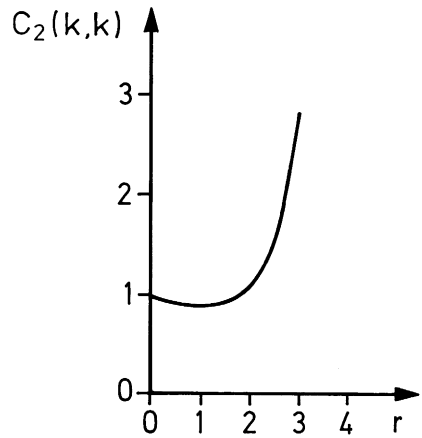

Fig. 2. Second-order correlation function $C_2(k,k) \equiv g^{(2)}$ as a function of the squeezing parameter r for pure squeezed states (from Ref. [27]).

The knowledge of $\mathscr{P}(\alpha)$ is (almost) equivalent to the knowledge of the density matrix. However in most cases the exact form of $\mathscr{P}(\alpha)$ is not accessible and one has to be content with certain approximations of it. Among these approximations the Gaussian form is privileged because:

(i) one can prove that $\mathscr{P}(\alpha)$ is of Gaussian form for a certain physical situation which is frequently met in many-body physics.

(ii) its use introduces an enormous mathematical simplification.

Proposition (i) is the subject of the central limit theorem which states that if

1. the number of sources becomes large;
2. they are stationary in the sense that their weight function $\mathscr{P}(\alpha)$ depends only on the absolute value $|\alpha|$;
3. they act independently,

then $\mathscr{P}(\alpha)$ is Gaussian. These conditions are known to be fulfilled in most cases of optics and presumably also in high-energy physics. Chaotic fields and in particular systems in thermal equilibrium are described by a Gaussian density matrix.

One of the reasons why the Gaussian form for $\mathscr{P}$ plays such an important part in correlation studies is the fact that for a Gaussian $\mathscr{P}(\alpha)$ all higher-order correlations can be expressed in terms of the first two correlation functions (see e.g. Refs. [3,5]).

On the other hand, the coherent state representation is particularly important for correlation studies because in this representation all correlation functions can be expressed in terms of the creation and annihilation operators $a^\dagger$ and a of the fields (particles). This follows from the Fourier expansion of an arbitrary field in second quantisation

$$\pi(x) = \sum_k [a_k \, \mathrm{e}^{-ikx} + a_k^\dagger \, \mathrm{e}^{ikx}] \; . \tag{2.21}$$

This property will be used extensively in Section 4.2 within the classical current formalism.

Correlation functions. The first-order correlation function reads

$$G^{(1)}(x,x') \equiv Tr[\rho\pi^{\dagger}(x)\pi(x')] \ . \tag{2.22}$$

Higher (nth)-order correlation functions are defined analogously by

$$G^{(n)}(x_1 \dots x_n, x_{n+1} \dots x_{2n}) \equiv Tr[\rho\pi^{\dagger}(x_1) \dots \pi^{\dagger}(x_n)\pi(x_{n+1}) \dots \pi(x_{2n})] \ . \tag{2.23}$$

In quantum field theory because of the mathematical complexity of the problem exact solutions of the field equations are available only in special cases. One such case will be discussed later on. However for strong interactions[9] even for this case one has to use phenomenological parametrisations of the correlation functions and determine the parameters (which have a definite physical meaning) by comparing with experiment.

In optics for stationary chaotic fields two particular parametrisations are used:

1. Lorentzian spectrum:

$$G^{(1)}(x_1, x_2) = \langle n_{\mathrm{ch}} \rangle \, \mathrm{e}^{-|x_1 - x_2|/\xi} \ . \tag{2.24}$$

2. Gaussian spectrum:

$$G^{(1)}(x_1, x_2) = \langle n_{\mathrm{ch}} \rangle \, \mathrm{e}^{-|x_1 - x_2|^2/\xi^2} \ , \tag{2.25}$$

ξ is the *coherence length* in x-space and $\langle n_{\mathrm{ch}} \rangle$ is the mean number of particles associated with the chaotic fields.

In [25] it has been proposed to use the analogy between time and rapidity in applying the methods of quantum optics to particle physics. Indeed in optics processes are usually stationary in time while in particle physics the corresponding stationary variable (in the rapidity plateau region) is rapidity.[10]

Pure coherent or pure chaotic fields are just extreme cases. In general one expects *partial coherence*, i.e. a superposition of coherent and chaotic fields

$$\pi = \pi_{\mathrm{coherent}} + \pi_{\mathrm{chaotic}} \ . \tag{2.26}$$

This leads for the Lorentzian case, e.g. to a second-order correlation function of the form

$$C_2(x,x') = 1 + 2p(1-p)\,\mathrm{e}^{-|x-x'|/\xi} + p^2\,\mathrm{e}^{-2|x-x'|/\xi} \ , \tag{2.27}$$

where p is the chaoticity, which varies between 0 (for purely coherent sources) and 1 (for totally chaotic sources). Eq. (2.8) is seen to be a particular form of the above equation for $\lambda = p = 1$ and it is clear herefrom that λ does not describe (partial) coherence as its name would imply. The presence of coherence introduces a new term into C_2. However, it is remarkable that the number of free parameters in Eq. (2.27) is the same as in Eq. (2.8). Formally, it appears as if there would act two

[9] By strong interactions we mean here interactions for which perturbative methods are inapplicable. They are present not only in hadronic physics but also in quantum optics.

[10] This property does not hold for other variables.

sources rather than one, but the "weights" and the space–time characteristics of these two sources are in a well-defined relationship.

This circumstance had been forgotten up to 1989 [28] both by experimentalists and theorists. The reason for this is the fact that during the 1980s the wave-function formalism was dominating the BEC literature, especially the experimental one.

From the foregoing discussion it should be clear that there are various reasons besides coherence why the bunching effect in BEC is reduced. However, it should also be clear that the empirical description of this state of affairs through the λ factor is possible only for totally chaotic sources. Since in an experiment this is never known a priori, this implies that the fitting of data with a formula of this type is misleading and should be avoided, the more so because the correct formula (2.27) does not contain more free parameters than Eq. (2.8). In the example presented above these free parameters are p and ξ for Eq. (2.27) and λ and the effective radius (which enters in $\tilde{f}$) for Eq. (2.8), respectively.

Using the rapidity–time analogy of Ref. [25] for a partially chaotic field the second-order correlation function in rapidity is given by Eq. (2.27), with x being replaced by rapidity y.

The fact that the last two terms in Eq. (2.27) are in a well-defined relationship and depend in a characteristic way on the two parameters p and ξ is a consequence of the superposition of the two *fields* (coherent and chaotic) and distinguishes a partially coherent source from a source which is a superposition of two independent *chaotic intensities*. Because of this the form (2.27) was proposed in [28] to be used as a signal for detection of coherence in BEC.[11] As a matter of fact, an attempt in this direction was made in an experimental study by Kulka and Lörstad [29]. In this analysis BEC data from pp and $\bar{p}p$ reactions at $\sqrt{s} = 53$ GeV were used to compare various forms of correlation functions. Among other things one considered formulae of QO type for rapidity

$$C_2 = 1 + 2p(1-p)\,\mathrm{e}^{-|y_1 - y_2|/\xi} + p^2\,\mathrm{e}^{-2|y_1 - y_2|/\xi} \tag{2.28}$$

(corresponding to a Lorentzian spectrum) and

$$C_2 = 1 + 2p(1-p)\,\mathrm{e}^{-|y_1 - y_2|^2/\xi^2} + p^2\,\mathrm{e}^{-2|y_1 - y_2|^2/\xi^2} \tag{2.29}$$

(corresponding to a Gaussian spectrum) as well as arbitrary superpositions of two chaotic sources of exponential or Gaussian form, respectively.

$$C_2 = 1 + \lambda_1\,\mathrm{e}^{-|y_1 - y_2|/\xi} + \lambda_2\,\mathrm{e}^{-2|y_1 - y_2|/\xi}\,, \tag{2.30}$$

$$C_2 = 1 + \lambda_1\,\mathrm{e}^{-|y_1 - y_2|^2/\xi^2} + \lambda_2\,\mathrm{e}^{-2|y_1 - y_2|^2/\xi^2}\,. \tag{2.31}$$

Here λ_1 and λ_2 represent arbitrary weights of the two chaotic sources.

Because of the limited statistics no conclusion could be drawn as to the preference of the QO form versus the two-source form. Similar inconclusive results were obtained when one replaced $y_1 - y_2$ in the above equations by the invariant momentum difference $Q^2 = (k_1 - k_2)^2$.

[11] Eq. (2.27) is a special case of superposition of coherent and chaotic fields; it can be considered as corresponding to point-like coherent and chaotic sources and a momentum independent chaoticity; superpositions of more general finite-size sources are considered in Section 4.4 (Eq. (4.40)) and Section 4.8.

2.2.1. Higher-order correlations

We have mentioned above that a characteristic property of the Gaussian form of density matrix (not to be confused with the Gaussian form of the correlator or the Gaussian form of the space–time distribution) is the fact that all higher-order correlation functions are determined just by the first two correlation functions. Since all BEC studies in particle physics performed so far assume a Gaussian density matrix, the reader may wonder why it is necessary to measure higher-order correlation functions.

There are at least three reasons for this:

(i) The conditions of the applicability of the above theorem and in particular the postulate that the number of sources is infinite and that they act independently can never be fulfilled exactly.

(ii) In the absence of a theory which determines from first principles the first two correlation functions, models for these quantities are used, which are only approximations. The errors introduced by these phenomenological parametrisations manifest themselves differently in each order and thus violate the above theorem even if (i) would not apply. Moreover, for certain parametrisations of the correlation functions the phases of the chaotic and coherent amplitudes disappear from the second-order correlation function (see Section 4.8) and are present only in higher-order correlation functions.

(iii) In experiments, because of limited statistics and sometimes also because of theoretical biases not all physical observables are determined, but rather averages over certain variables are performed, which again introduce errors which propagate (and are amplified) from lower-to-higher correlations.

Conversely, by comparing correlation functions of different order, one can test the applicability of the theorem quoted above and pin down more precisely the parameters which determine the first two correlation functions (e.g. the chaoticity p and the correlation length ξ in Eqs. (2.28) and (2.29)), which is essentially the purpose of particle interferometry.

The phenomenological application of these considerations will be discussed in the following as well as in Section 6.3 for the particular case of the invariant Q variable, but arguments (i)–(iii) have general validity. It would be a worthwhile research project to compare the deviations introduced in the relation between lower and higher correlation functions, due to (i) with those introduced by (ii) and (iii).

The simplification brought by the variable Q can be enhanced by a further approximation proposed by Biyajima et al. [30]. With the notation $Q_{ij} = k_i - k_j$ the analogue of Eq. (2.29) can be written

$$C_2 = 1 + 2p(1-p)\exp(-R^2Q_{12}^2) + p^2\exp(-2Q_{12}^2R^2) . \tag{2.32}$$

For the third-order correlation function one obtains

$$\begin{aligned} C_3 = 1 &+ 2p(1-p)[\exp\{-R^2Q_{12}^2\} + \exp\{-R^2Q_{13}^2\} + \exp\{-R^2Q_{23}^2\}] \\ &+ p^2[\exp\{-2R^2Q_{12}^2\} + \exp\{-2R^2Q_{13}^2\} + \exp\{-2R^2Q_{23}^2\}] \\ &+ 2p^2(1-p)[\exp\{-R^2(Q_{12}^2 + Q_{23}^2)\} + \exp\{-R^2(Q_{13}^2 + Q_{23}^2)\} \\ &+ \exp\{-R^2(Q_{12}^2 + Q_{13}^2)\}] + 2p^3[\exp\{-R^2(Q_{12}^2 + Q_{13}^2 + Q_{23}^2)\}] . \end{aligned} \tag{2.33}$$

Ref. [30] proposed to use symmetrical configurations for all two-particle momentum differences, i.e. to consider Q_{ij} independent of (i,j). For C_2 this assumption does not of course introduce any modifications. However for higher orders the simplification is important. Thus, e.g. for the third-order correlation with $Q_{12} = Q_{13} = Q_{23}$ and with the definition $Q^2_{\rm three} = Q^2_{12} + Q^2_{13} + Q^2_{23}$ Eq. (2.33) becomes

$$C_3 = 1 + 6p(1-p)\exp\{-\tfrac{1}{3}R^2Q^2_{\rm three}\} + 3p^2(3-2p)\exp\{-\tfrac{2}{3}R^2Q^2_{\rm three}\} + 2p^3\exp\{-R^2Q^2_{\rm three}\}\ . \tag{2.34}$$

In Ref. [30] similar expressions for C_4 and C_5, again for a Gaussian correlator, were given.

These relations for higher-order BEC were subjected to an experimental test in Ref. [31], using the UA1 data for $\bar{p}p$ reactions at $\sqrt{s} = 630$ and 900 GeV. For reasons which will become clear immediately, we discuss here this topic in some detail.

The procedure used in [31] for this test consisted in determining R and p separately for each order q of the correlation and comparing these values for different q. It was found that a Gaussian correlator did not fit the data. Next in [31], one tried to replace the Gaussian correlator by an exponential (see Eq. (2.28)). To do this one substituted simply in the expressions for the correlation functions of Ref. [30] the factor $\exp(-R^2Q^2)$ with $\exp(-RQ)$. Such a procedure was at hand given the fact that for C_2 the QO formulae both for an exponential correlator and a Gaussian correlator were known [28] and their comparison suggested just this substitution, as seen from Eq. (2.29).

In [31] one used then for the exponential correlator the relations

$$C_2^{\rm empirical} = 1 + 2p(1-p)\exp(-RQ_{12}) + p^2\exp(-2Q_{12}R)\ , \tag{2.35}$$

$$C_3^{\rm empirical} = 1 + 6p(1-p)\exp(-\tfrac{1}{3}RQ_{\rm three}) + 3p^2(3-2p)\exp(-\tfrac{2}{3}RQ_{\rm three}) + 2p^3\exp(-RQ_{\rm three})\ . \tag{2.36}$$

With these modified formulae one still could not find in [31] a unique set of values p and R for all orders of correlation functions. However now a clearer picture of the "disagreement" between the QO formalism and the data emerged. It seemed that while the parameter p was more or less independent of q, the radius R increased with the order q in a way which could be approximated by the relation

$$R_q = R\sqrt{\tfrac{1}{2}q(q-1)}\ . \tag{2.37}$$

However in [32] it was shown that the findings of [31] and in particular Eq. (2.37) not only did not contradict QS but on the contrary constituted a confirmation of it. While Eq. (2.35) for the second-order correlation function coincides with that derived in quantum optics for an exponential spectrum, this is not the case with the expressions for higher-order correlations $C_q^{\rm empirical}$ (Eq. (2.36)). The formulae for C_3, C_4 and C_5 corresponding in QS to an exponential correlator and derived in [32] differ from the empirical ones used in [31]. As an example we quote

$$C_3 = 1 + 6p(1-p)\exp(-(1/\sqrt{3})RQ_{\rm three}) + 3p^2(3-2p)\exp(-(2/\sqrt{3})RQ_{\rm three}) + 2p^3\exp(-\sqrt{3}RQ_{\rm three})\ . \tag{2.38}$$

As observed in [32] and as one easily can check by comparing Eq. (2.36) with Eq. (2.38), one can make the empirical formulae for C_q used in [31] coincide with the correct ones, by replacing the parameter R with a scaled parameter R_s and the relation between R and R_s, is nothing else but $R_s = R_q$, where R_q is given by Eq. (2.37). The fact that this happens for three different orders, i.e. for C_3, C_4 and C_5 makes a coincidence quite improbable.

By empirically modifying the formulae of higher-order BEC for the exponential case, paper [31] had explicitly violated QS and the "phenomenological" relationship (2.37) between R and q just compensated this violation.

The fact that this compensation and the final agreement between theory and experiment was not perfect is not surprising and is discussed in [32]. Besides reasons (i)–(iii) mentioned above, one has to take into account the fact that the QO formalism in which Eqs. (2.35) and (2.33) are based assumes stationarity in Q, i.e. assumes that the correlator depends only on the difference of momenta $k_1 - k_2$ and not also on their sum. As mentioned already this condition is in general not fulfilled in BEC. Furthermore, the parameter p, if it is related to chaoticity, is in general momentum dependent (see Section 4.8). Also, the symmetry assumption, $Q_{i,j}$ independent of i,j, may be too strong. Besides these theoretical caveats, there are also experimental problems, related to the fact that the UA1 experiment is not a dedicated BEC experiment and thus suffers from specific diseases, which are common to almost all particle physics BEC experiments performed so far. Among other things, there is no identification of particles (only 85% of the tracks recorded are pions), and the normalisation of correlation functions is the "conventional" one, i.e. not based on the single inclusive cross sections as the definition of correlation functions demands (see (2.7) and Section 4.11), but rather uses an empirically determined "background" ensemble.

In the mean time further theoretical and experimental developments took place.

On the theoretical side a new space–time approach to BEC was developed [3,33] which is more appropriate to particle physics and which contains as a special case the QO formalism. In particular, the two exponential features of the correlation function is recovered.

On the experimental side a new technique for the study of higher-order correlations was developed, the method of correlation integrals which was applied [34] to a subset of the UA1 data in order to test the above-quoted QO formalism. The fits were restricted to second- and third-order cumulants only. Again it was found that by extracting the parameters p and R from the second-order data, the "predicted" third-order correlation, this time by using a correct QO formula, differed significantly from the measured one.

If confirmed, such a result could indicate that the QO formalism provides only a rough description of the data and that higher precision data demand also more realistic theoretical tools. Such tools are the QS space–time approach to BEC presented in Section 4.3. A further, but more remote possibility would be to look for deviations from the Gaussian form of the density matrix. However, it seems premature to speculate along these lines given the fact that the procedure used to test the relation between the second- and third-order correlation functions has to be qualified. Indeed in [34] one did not perform a *simultaneous* fit of second- and third-order data to check the QO formalism. Such a simultaneous fit appears necessary before drawing conclusions, because as mentioned above (see (ii) and (iii)), the errors involved in "guessing" the form of the correlator, and the fact that the variable Q does not characterise completely the two-particle correlation, limit the applicability of the theorem which reduces higher-order correlations to first and second ones. As a matter of fact, it was found [35] (see also Section 6.3) in a comparison of the QS space–time approach with higher-order correlation data, that the second-order correlation data is quite insensitive to the values of the parameters which enter the correlator, while once higher-order data are used in a simultaneous fit, a strong delimitation of the acceptable parameter values results.

Thus there are several possible solutions if one restricts the fit to the second-order correlation and the correct one among these can be found only by fitting *simultaneously* all correlations. If by accident one chooses in a lower correlation the wrong parameter set, then the higher correlations cannot be fitted anymore.[12]

Before ending these phenomenological considerations in which the variable Q played a major part, a few remarks about its use may be in order.

The invariant momentum difference Q. BEC studies in particle physics use often a privileged variable namely the squared momentum difference:

$$Q^2 = (k_1 - k_2)^2 = (\boldsymbol{k}_1 - \boldsymbol{k}_2)^2 - (E_1 - E_2)^2 \ . \tag{2.39}$$

It owes its special role to the fact that it is a relativistic invariant and it has already been used in the pioneering paper by GGLP [8]. It also has the advantage that it involves all four components of the momenta k simultaneously so that the intercept of the correlation function $C_2(\boldsymbol{k},\boldsymbol{k})$ coincides with $C_2(Q = 0)$. Thus by measuring C_2 as a function of one single scalar quantity Q one gets automatically the intercept. This is not the case with other single scalar quantities used in BEC like $y_1 - y_2$ or $k_{\perp,1} - k_{\perp,2}$ which characterise the intercept only partially.

On the other hand, Q suffers from certain serious diseases which make its use for practical interferometrical purposes questionable.

The first and most important deficiency of Q is the fact that it mixes time and space coordinates: the associated quantity R in the conventional parametrisation of the correlation function $C_2 = 1 + \lambda \exp(-R^2Q^2)$ is neither a radius nor a lifetime, but a combination of these, which cannot be easily disentangled. Another deficiency of Q, which is common to all single scalar quantities is the circumstance that it does not fully characterise the correlation function. Indeed, the second-order correlation function C_2 is in general a function of six independent quantities which cannot be replaced by a single variable.

An improvement on Q was proposed by Cramer [36] with the introduction of *coalescence* variables which constitute a set of three boost invariant variables to replace for, a pair of identical particles, with the single variable Q. They are related to Q by $Q^2 = 2m^2(C_\mathrm{L} + C_\mathrm{T}^2 + C_\mathrm{R}^2)$, where $C_\mathrm{L}, C_\mathrm{T}, C_\mathrm{R}$ denote longitudinal, transverse and radial coalescence respectively, and m the mass of the particle. They have the properties that $C_\mathrm{L} = 0$ when $y_1 = y_2$, $C_\mathrm{T} = 0$ when $\phi_1 = \phi_2$ and $C_\mathrm{R} = 0$ when either $m_1 = m_2$ or $k_1 = k_2$. Here m_i is the transverse mass and ϕ the azimuthal angle in the transverse plane. It is shown in [36] that with these new variables a Lorentz invariant separation of the space- and time-like characteristics of the source is possible, within the kinematical assumptions involved by the particular choice of the coalescence variables. This separation is however rather involved. In [36] the coalescence variables are used for the introduction of Coulomb corrections into second- and higher-order correlation functions.

Another way to compensate in part for the fact that one single variable does not characterise completely the two-particle system, but which maintains the use of Q is, as explained above, to consider higher-order correlations.

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[12] The fact that in [34] the correlations were normalised by mixing events rather than by comparing with the single inclusive cross sections in the same event, as prescribed by the definition of the correlation functions may also influence the applicability of the central limit theorem.

3. Final state interactions of hadronic bosons

One of the most important differences between the HBT effect in optics and the corresponding GGLP effect in particle physics is that in the first case we deal with photons while in the second case with hadrons.

While photons in a first approximation do not interact, hadrons do interact. This interaction has two effects: (i) it influences the correlation between identical hadrons and (ii) it leads to correlations also between non-identical hadrons.

This review deals only with correlations due to the identity of particles and in particular with Bose–Einstein correlations. Therefore only effect (i) will be discussed.[13]

Effect (i) is usually described in terms of final state interactions. In some theoretical studies (see e.g. [38] and references quoted there) emission of particles at different times is also treated as an effective final state interaction.

From the BEC point of view the final states interaction constitutes in general an unwanted background, which has to be subtracted in order to obtain the "true" quantum statistical effect on which interferometry measurements are based. That this is not always a trivial task will be shown in the following.

There are two types of final state interactions in hadronic interferometry: electromagnetic, traded under the generic name of Coulomb interactions, and strong. Furthermore, one distinguishes between one-body final state interactions and many-body final state interactions.

3.1. Electromagnetic final state interactions

The plane wave two-body function used in the considerations above (see Eq. (2.5)) applies of course only for non-interacting particles.

As a first step towards a more general treatment consider charged particle interferometry. As a matter of fact the vast majority of BEC studies, both of experimental and theoretical nature, refer to charged pions. For two-particle correlations, we will have to consider the interaction of each member of a pair with the charge of the source and the Coulomb interaction between the two particles constituting the pair. The first effect will affect primarily the single-particle probabilities and is not expected to depend on the momentum difference q. Attention has been paid so far mostly to the second effect, i.e. the modification of the two-particle wave function due to the Coulomb interaction between the two particles. While initially, having in mind the Gamow formula, it was assumed that this effect is (for small q values) quite important, at present serious doubts about these estimates have arisen. The model dependence of corrections for this effect makes it almost imperative that experimental data should be presented also without Coulomb corrections, so that it should be left to the reader the possibility of introducing (or not introducing) corrections according to her/his own prejudice.

[13] The reader interested in correlations between non-identical particles is referred to [2] for the period up to 1990; for more recent literature see [37], where correlations in low energy heavy ion reactions are reviewed.

3.1.1. Coulomb correction and "overcorrection"

As usual one separates the centre-of-mass motion from the relative motion. For the last one the scattering wave function reads

$$u(\boldsymbol{r}) \to_{r\to\infty} \mathrm{e}^{ikz} + r^{-1} g(\theta,\phi)\mathrm{e}^{ikr} , \tag{3.40}$$

where the relative position vector $\boldsymbol{r}$ has polar coordinates r, θ, ϕ.

The form of the function g depends on the scattering potential. In the Coulomb case the corresponding Schrödinger equation can be solved exactly and the correction to the two-particle wave function and the correlation function can be calculated. So far at least three sophisticated procedures have been used for this purpose. In [39] the value of the square modulus of the wave function u in the origin $r = 0$ was proposed as a correction term G to the correlation function C_2. Up to non-interesting factors this is the Gamow factor which reads

$$|u(0)|^2 = 2\pi\eta/(\exp(2\pi\eta) - 1) = G(\eta) . \tag{3.41}$$

Here

$$\eta = \alpha m_\pi / q \tag{3.42}$$

and $q = |k_1 - k_2|$. m_π is the pion mass.

However as pointed out by Bowler [40] and subsequently also by others, in BEC this approximation may be questionable. Indeed in a typical e^+–e^- reaction, e.g. the source which gives rise to BEC has a size of the order of 1 fm which is a large number compared with a typical "Coulomb length" r_t defined as the classical turning point where the kinetic energy balances the potential Coulomb energy:

$$\tilde{q}^2/2m_{\mathrm{red}} = e^2/r_t , \tag{3.43}$$

where $\tilde{q} = q/2$ and m_{red} is the reduced mass of the pion pair. For a typical BEC momentum difference of $q = 100$ MeV, one gets from (3.43) $r_t = 0.08$ fm. This suggests that by taking the value of the wave function at $r = 0$ one overestimates the Coulomb correction.

More recently, Biyajima and collaborators [41] (see also [42]) have considered a further correction to the correction proposed by Bowler, which decreases even more the Coulomb effect and which is also of heuristic interest. In [41] it is pointed out that the wave function (3.40) used by Bowler has not taken into account the symmetry of the two-particle system. It has to be supplemented by an exchange term so that the r.h.s. of (3.40) becomes

$$(\mathrm{e}^{iqz} + \mathrm{e}^{-iqz}) + [f(\theta,\phi) + f(\pi - \theta, \phi + \pi)]r^{-1}\mathrm{e}^{iqr} . \tag{3.44}$$

The corrections due to this new effect are of the same order as those found in [40] and go in the same direction.

Another approach to the Coulomb correction in BEC has been suggested by Baym and Braun-Munzinger [43]. Starting from the observation of [40] about the classical turning point these authors propose the use for heavy ion reactions of a *classical* Coulomb correction factor arising from the assumption that the Coulomb effect of the pair is negligible for separations less than an initial radius r_0. This model is tested by comparing its results with experimental data on

$\pi^+\pi^-, \pi^-p, \pi^+p$ correlations in heavy ion collisions[14] (Au–Au at AGS energies). The assumption behind this comparison is that the observed correlations are *solely* due to the Coulomb effect.[15] Indeed qualitatively this seems to be the case: thus the data for the $\pi^+\pi^-$ and π^-p correlations show a bunching effect characteristic of an attractive interaction while the data for the π^+p correlation show an antibunching effect, characteristic of repulsion. After this test the authors compare their correction with the Gamow correction and find that the last one is much stronger. Wherefrom they also conclude that the Gamow factor overestimates quite appreciably the Coulomb effect.[16]

An even stronger conclusion is reached by Merlitz and Pelte [47] from the solution of the time-dependent Schrödinger equation for two identical charged scalar bosons in terms of wave packets. These authors find that the "expected" Coulomb effect in the correlation function is obliterated by the dispersion of the localised states and is thus unobservable. This makes the interpretation of experimentally observed $\pi^+\pi^-$ correlations in terms of Coulomb effects even more doubtful.

The theoretical studies of the Coulomb effect in BEC quoted so far are based on the solution of the Schrödinger equation and apply in fact only for the non-relativistic case. While one might argue that the relative motion of two mesons in BEC is for small q non-relativistic, this is not true for the single-particle distributions (see below). Therefore, in principle, one should replace the solution of the Schrödinger equation in the Coulomb field used above by the corresponding solution of the Klein–Gordon equation. This apparently has not yet been done, with the exception of of a calculation by Barz [48] who investigated the influence of the Coulomb correction on the measured values of radii. He found an important change of these radii due to the Coulomb field only for momenta ≤ 200 MeV.

Finally, one must mention that the corrections of the wave function described above do not take into account the fact that the charge distribution of a meson is not point like, but has a finite extension of the order of 1 fm. This means that in principle the Schrödinger (or the Klein–Gordon) equation has to be solved with a Coulomb potential modified by this finite size effect. Given the great sensitivity of BEC on small corrections in the wave function, this might be a worthwhile enterprise for future research. As a matter of fact it is known from atomic physics (isotopic and isomeric shifts and hyperfine structure) that these finite size effects lead to observable consequences.

Besides the wave function effect which influences the BEC due to directly produced particles or those originating from short-lived resonances, one has to consider [40] the Coulomb overcorrection applied to pairs of which one particle is a daughter of a long-lived state. This effect may bias the correction by up to 20%.

Coulomb correction for higher-order correlations. The Coulomb corrections discussed above were limited to single- and two-body interactions. In present high-energy heavy ion reactions we have

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[14] The measured $\pi^+\pi^-$ correlations were also used in two recent experimental papers [44,45] to estimate the Coulomb correction. Why such a procedure is questionable is explained below.

[15] This assumption has to be qualified among other things because the final state strong interactions effects due to resonances can also influence these correlations. Furthermore there also exists a quantum statistical correlation for the $\pi^+\pi^-$ system (see Sections 4.4, 4.7 and 4.8) which, however may be weak.

[16] See however also Ref. [46] where rescattering is added to the classical Coulomb effect and where somewhat different results are obtained. It is unclear whether the strong position-momentum correlations implicit in this rescattering model do not violate quantum mechanics.

already events with hundreds of particles and in the very near future at the relativistic heavy ion collider at Brookhaven (RHIC) this number will increase by an order of magnitude. Then many-body final state interactions may become important. Unfortunately, the theory of many-body interactions even for such a "simple" potential as the Coulomb one is apparently still unmanageable. The long-range nature of the electromagnetic interaction does not make this task simpler. Bowler [40] sketched a scenario for Coulomb screening based on the string model. In such a model particles are ordered in space–time, so that, e.g. at least one π^+ must be situated between the members of a $\pi^-\pi^-$ pair. While the net influence of this effect on $\pi^+\pi^-$ is expected to be small, for a $\pi^-\pi^-$ pair the situation is different, because instead of repulsion one obtains attraction. In the case of long-range interactions such an effect may become important if the π_1^+ propagates together with the $\pi_2^-\pi_3^-$ pair. This happens if $Q_{12} \sim Q_{13} \sim Q_{23}$. To take care of this effect Bowler suggests the replacement

$$C(Q_{23}) \to C(Q_{23})\langle C^{-1}(Q_{12})C^{-1}(Q_{13})\rangle_{k3} \tag{3.45}$$

for the Coulomb $\pi^-\pi^-$ correction and

$$C(Q_{12}) \to C(Q_{12})\langle C^{-1}(Q_{23})C(Q_{13})\rangle_{k3} \tag{3.46}$$

for the Coulomb $\pi^+\pi^-$ correction. Here $\langle \dots \rangle_{k_3}$ symbolises averaging with respect to the momentum of particle 3. While on the average

$$\langle C^{-1}(Q_{23})C(Q_{13})\rangle_{k3} \tag{3.47}$$

is unity, at small Q the function $C(Q)$ oscillates rapidly and therefore the factor

$$\langle C^{-1}(Q_{12})C^{-1}(Q_{13})\rangle_{k3} \tag{3.48}$$

is sensitive to the distribution of Q_{12}, Q_{13} associated with the local source. According to [40] this last factor for $\pi^+\pi^-$ pairs does not exceed 0.5% but for like-sign pairs no estimate is provided.

Coulomb and resonance effect in single inclusive cross sections. At a first look one might be tempted to believe that for single inclusive cross sections in heavy ion reactions the estimate of Coulomb effects is straightforward. Unfortunately, this is not the case and so far there is no reliable theoretical estimate of this effect. This is so because the produced charged secondaries do not move simply in the electromagnetic field of the colliding nuclei but at the same time interact with all the other secondaries. In view of this situation, recently an attempt has been made to put in evidence experimentally the Coulomb effect in the single inclusive cross section of pions in heavy ion reactions [49]. In this experiment an excess of negative pions over positive pions in Pb–Pb reactions at 158 AGeV was observed which the authors of [49] attributed to the Coulomb interaction of produced pions with the nuclear fireball. However, this interpretation has been challenged in [50] where, in a detailed hydrodynamical simulation it was shown that a similar excess in the π^-/π^+ ratio is expected as a consequence of resonance (especially hyperon) decays. This qualification goes in the same direction as that mentioned above with respect to the exaggeration of the effects of Coulomb interaction in BEC. It also illustrates the complexity of the many-body problem of heavy ion reactions even for weak interactions like the electromagnetic one, which in principle are well known.

3.2. Strong final state interactions

This is a very complex problem because we are dealing with non-perturbative aspects of quantum chromodynamics. That there is no fully satisfactory solution to this problem can be seen from the very fact that we have at least three different approaches to it. As will become clear from the following, these different approaches must not be used simultaneously, as this would constitute double (or triple) counting which is also why a fourth solution proposed here which is of heuristic nature will appear in many cases more appealing and more efficient.

3.2.1. Final state interactions through resonances

The majority of secondaries produced in high-energy collisions are pions out of which a large fraction (between 40% and 80%) arises from resonances.[17] Since the resonances have finite lifetimes and momenta, their decay products are created in general outside the production region of the "direct" pions (i.e., pions produced directly from the source) and that of the resonances. As a consequence, the two-particle correlation function of pions reflects not only the geometry of the (primary) source but also the momentum spectra and lifetimes of resonances [2]. Kaons are much less affected by this circumstance [52]; however, correlation experiments with kaons are much more difficult because of the low statistics. For a more detailed discussion of kaon BEC see Section 5.1.3.

The (known) resonances have been taken into account explicitly within the wave-function formalism (see e.g. [17,53,18]), the string model [22,23] or within other variants of the Wigner function formalism (see e.g. [19,54–56]). The drawback of this explicit approach is that it is rather complicated, it is usually applicable only at small momenta differences q and it presupposes a detailed knowledge of resonance characteristics, including their weights, which, with few exceptions, cannot be measured directly and have to be obtained from event generators[18] or other models.[19]

With these essential caveats in mind one finds that the distortion of the two-particle correlation function due to resonance decay leads to two obvious effects: (a) the effective radius of the source increases, i.e., the width of the correlation function decreases, and (b) due to the finite experimental resolution in the momentum difference the presence of very long-lived resonances leads to an apparent decrease of the intercept of the correlation function. Effect (b) is particularly important if one wants to draw conclusions from the intercept about a possible contribution of a coherent component in multiparticle production.

In hydrodynamical studies of multiparticle production processes one considers resonances within the Wigner function approach (see Section 5.1.3).

3.2.2. Density matrix approach

In Ref. [59] one describes the effect of strong final state interactions by constructing a density matrix based on an effective Lagrangian of Landau–Ginzburg form as used in statistical physics

[17] For experimental estimates, see Ref. [51].

[18] That event generators are not a reliable source of information for this purpose was demonstrated in the case of e^+–e^- reactions in [57,58].

[19] In Ref. [56] the weights were determined from thermodynamical considerations.

and quantum optics (see also [60]). This is a more theoretical approach as it allows to study the effect of the strength of the interaction g on BEC, albeit in an effective Lagrangian description.

One writes the density matrix

$$\rho = Z^{-1}\int \delta\pi |\pi\rangle e^{-F(\pi)}\langle\pi| , \qquad Z = \int \delta\pi e^{-F(\pi)} , \tag{3.49}$$

where F is the analogue of the Landau–Ginzburg free energy and the integrals are functional integrals over the field π. This field is written as a superposition of coherent π_c and chaotic fields π_{ch}:

$$\pi = \pi_c(y) + \pi_{ch}(y) . \tag{3.50}$$

The variable y refers in particular to rapidity. The total mean multiplicity $\langle n\rangle$ is related to the field π by

$$\langle n\rangle = \langle|\pi^2|\rangle . \tag{3.51}$$

Similar relations hold for the coherent and chaotic parts of π.

One assumes stationarity in y, i.e. the field correlator $G(y, y') = \langle\pi(y)\pi(y')\rangle$ depends only on the difference $y - y' \equiv \Delta y$. One writes the Landau–Ginzburg form for F as

$$F(\pi) = \int_0^y dy\left[a\pi(y) + b\left|\frac{\partial\pi(y)}{\partial y}\right|^2 + g|\pi(y)|^4 \right] , \tag{3.52}$$

where a, b and g are constants. The strong interaction coupling is represented by g. The constants a, b can be expressed in terms of $\langle n_{ch}\rangle$ and the "coherence length" ξ which is defined through the correlator G (see Eq. (2.24)).

The main result of these rather involved calculations is that while the interaction does not play any significant role in the value of the intercept $C_2(0)$ it plays an important part in ($C_2(\Delta y \neq 0)$). This situation is illustrated in Figs. 3 and 4. Thus it is seen in Fig. 3 that all C_2 curves for various g coincide in the origin. The effect of the interaction in this approach is similar to that of (short-lived) resonances, i.e. it leads to a decrease of the width of the correlation function. The sensitivity of C_2 on g suggests that the shape of the correlation function can, in principle, be used for the experimental determination of g. One finds furthermore that there is no g-dependence for purely chaotic or purely coherent sources. This observation suggests that for a strongly coherent or chaotic field the final state interaction does not manage to disturb the correlation.

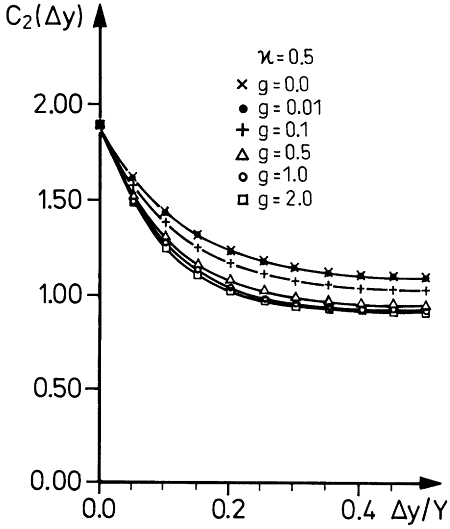
Note that the curves in Fig. 4 intersect at some Δy which depends on the chaoticity. This is characteristic for correlations treated by quantum statistics, when one has a superposition of coherent and chaotic fields, and is a manifestation of the appearance of two (different) functions in the correlation function.

We recall (see Section 2.2) that for the particular case of a Lorentzian spectrum one gets, *in the absence of final state interactions*,

$$C_2(y, y') = 1 + 2p(1-p)e^{-|y-y'|/\xi} + p^2 e^{-2|y-y'|/\xi} \tag{3.53}$$

instead of the empirical relation

$$C_2(y, y') = 1 + \lambda \exp(-|y-y'|/\xi) . \tag{3.54}$$



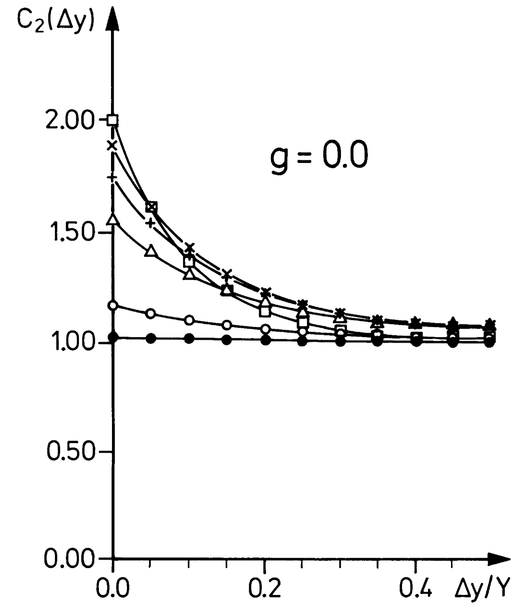


Fig. 3. Second-order correlation function as given by the quantum statistical formalism of Ref. [59] for various values of the coupling constant g for $\kappa = 0.5$ (from [59]). The parameter κ is related to the chaoticity p via the relation $\kappa = \langle n_c \rangle / \langle n_{ch} \rangle = 1/p - 1$ where $p = \langle n_{ch} \rangle / \langle n \rangle$ and $n = n_c + n_{ch}$ is the total multiplicity. Y is the maximum rapidity.

Fig. 4. The same as in Fig. 3 for various values of κ at $g = 0$ (from [59]).

As shown in Fig. 5 the parametrisation (3.54) leads to parallel curves for various values of λ, while the more correct parametrisation of Ref. [59] leads to intersecting curves. This is due to the fact that the relation for C_2 derived in [59] contains as a particular case (for $g = 0$) Eq. (3.53) and retains the essential feature of Eq. (3.53), which consists in the superposition of two exponentials.

3.2.3. Phase shifts

For charged pions the strong final state interactions can also be described by phase shifts. It is known that for an isospin $I = 2$ state (this is the isospin of a system of two identically charged pions) the corresponding strong interaction is repulsive. However it has been suggested [61] that the range of strong interactions is smaller than the size of the hadronic source and therefore the correlation should be essentially unaffected by this effect.[20] Even for particle reactions like hadron–hadron or e^+–e^- the size of the source is of the order of 1 fm while the range of interaction is only 0.2 fm. On the other hand, the effective size quoted above arises because of the joint contribution to BEC of direct pairs and resonances. So it is interesting to analyse these two contributions separately.

[20] The separation between resonances and phase shifts is of course not rigorous because phase shifts reflect also the effect of resonances; however as long as phase shifts constitute a small effect, this should not matter.

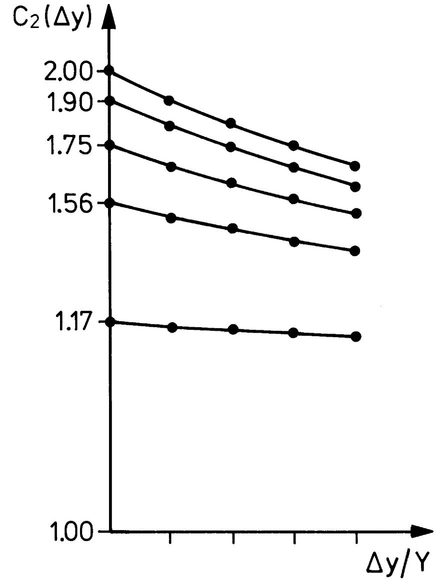


Fig. 5. Second-order correlation function for various values of the "incoherence" parameter λ as given by the phenomenological equation (3.54).

Two identically charged pions are practically always produced together with a third pion of opposite charge. Then according to Bowler [61] one has also to consider the $I = 0$ attractive interaction between the oppositely charged pions and this compensates largely for the $I = 2$ state interaction. Thus it appears that also in particle reactions only resonances play an important role in final state strong interactions.

The considerations about final state interactions made in this subsection treat separately Coulomb and strong interactions. This is permitted as long as we deal with small effects or when the ranges of the two types of interactions do not overlap. For very small distances this is not anymore the case. Furthermore, the Gamow and the phase shift corrections are based on the wave-function formalism which ignores the possibility of creating particles. However, when entering the non-classical region the well-known difficulties of the wave-function formalism become visible (Klein paradox). To consider this effect, in Ref. [62] the joint contribution of the strong interaction potential and the Coulomb potential are analysed in a version of the Bethe–Salpeter equation for spinless particles. It is found that as expected also from the considerations presented above the strong interaction diminishes appreciably the Gamow correction.

3.2.4. *Effective currents*

As will be explained below the most satisfactory approach to BEC is at present the classical current approach, based on quantum field theory. Here three types of source characteristics appear: the chaoticity, the correlation lengths/times and the space–time dimensions. It is obvious that these quantities already contain information about the nature of the interaction and therefore it is quite natural to consider them as effective parameters which describe all the effects of strong final state interactions. This suggests that rather than starting with "bare" non-interacting currents and introducing afterwards final state interactions, it might be preferable to assume from the beginning that the currents as defined by the field theoretical formalism are "effective" and therefore already contain all final state interactions.

This approach to strong final state interactions is probably the most recommendable one at the present stage for BEC in reactions induced by hadrons or leptons because it: (1) is simpler; (2) avoids double counting; (3) avoids the use of poorly known resonance characteristics; (4) avoids the use of the ill-defined concept of final state interactions for strong interactions. For heavy ion reactions, when hydrodynamical methods are used the explicit consideration of resonances can be practised up to a certain point without major difficulties and then the strong final state interactions can be taken into account through these resonances (see Section 5.1.3).

To conclude this discussion of final state interactions in BEC, it is interesting to note that in boson condensates the final state interactions might be different than in normal hadronic sources. In a condensate the Bose field becomes long range in configuration space. This can be understood as a consequence of the fact that in a condensate the effective mass of the field carrier vanishes. Indeed a calculation [63] based on the chiral sigma model shows that the effective range of the pion field can increase several times due to this effect.

4. Currents

4.1. Classical versus quantum currents

In Section 2.2 we were concerned mainly with the properties of fields and did not ask the question where these fields come from. In the present section we shall pose this question and try to answer it.

We start by recalling the definition of correlation functions within quantum field theory.

Let $a_i^\dagger(\boldsymbol{k})$ and $a_i(\boldsymbol{k})$ be the creation and annihilation operator of a particle of momentum $\boldsymbol{k}$, where the index i labels internal degrees of freedom such as spin, isospin, strangeness, etc. The n-particle inclusive distribution is

$$\frac{1}{\sigma}\frac{\mathrm{d}^n\sigma^{i_1\dots i_n}}{\mathrm{d}\omega_1\dots\mathrm{d}\omega_n} = (2\pi)^{3n}\prod_{j=1}^{n} 2E_j\; Tr(\rho_\mathrm{f}\, a_{i_1}^\dagger(\boldsymbol{k}_1)\dots a_{i_n}^\dagger(\boldsymbol{k}_n)a_{i_n}(\boldsymbol{k}_n)\dots a_{i_1}(\boldsymbol{k}_1))\ , \tag{4.1}$$

where

$$\mathrm{d}\omega_\mathrm{i} = \mathrm{d}^3k_\mathrm{i}/(2\pi)^3 2E_\mathrm{i} \tag{4.2}$$

is the invariant volume element in momentum space. With the notation

$$G_n^{i_1\dots i_n}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n) \equiv (1/\sigma)\frac{\mathrm{d}^n\sigma^{i_1\dots i_n}}{\mathrm{d}\omega_1\dots\mathrm{d}\omega_n}\ , \tag{4.3}$$

the general n-particle correlation function is defined as

$$C_n^{i_1\dots i_n}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n) = G_n^{i_1\dots i_n}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n)/G_1^{i_1}(\boldsymbol{k}_1)\cdot\ \dots\ \cdot G_n^{i_n}(\boldsymbol{k}_n)\ . \tag{4.4}$$

The particles which the operators $a^\dagger$ and a are associated with are the quanta of the field (which we denote in general by ϕ); these particles as well as the density matrix ρ refer to the final state where measurements take place. On the other hand, we usually know (or guess) the density matrix only in the initial state. Therefore, we have to transform the above expression so that eventually the density matrix in the final state ρ_f is replaced by the density matrix in the initial state ρ_i, while the fields will continue to refer to the final state. To emphasize this we wrote in Eq. (4.1) ρ_f. We have

$$\rho_\mathrm{f} = S\rho_\mathrm{i}S^\dagger \ , \tag{4.5}$$

so that

$$G_1(\boldsymbol{k}) = (2\pi)(2E_1)\,Tr\{\rho_\mathrm{i}S^\dagger a^\dagger(\boldsymbol{k})a(\boldsymbol{k})S\} \ , \tag{4.6}$$

$$G_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = (2\pi)^6(2E_1)(2E_2)\,Tr\{\rho_\mathrm{i}S^\dagger a^\dagger(\boldsymbol{k}_2)a^\dagger(\boldsymbol{k}_1)a(\boldsymbol{k}_1)a(\boldsymbol{k}_2)S\} \ . \tag{4.7}$$

Thus, if the initial conditions i.e. ρ_i are given, in principle the knowledge of the S matrix suffices to calculate the physical quantities of interest. In one case the S matrix can even be derived without approximations. This happens when the currents are classical and we shall discuss this case in some detail in this and the following section.

Before doing this, we shall consider briefly the more general case when the currents are not necessarily classical.

The S matrix is given by the relation

$$S = \mathscr{T}\exp\left\{\mathrm{i}\int \mathrm{d}^4x L_\mathrm{int}(x)\right\} \ , \tag{4.8}$$

where the interaction lagrangian L_int is a functional of the fields ϕ. $\mathscr{T}$ is the chronological time-ordering operator; we shall use below also the antichronological time-operator $\tilde{\mathscr{T}}$.

Consider for simplicity a scalar field produced by a current J. Then

$$L_\mathrm{int}(x) \equiv J(x)\phi(x) \ . \tag{4.9}$$

Eqs. (4.8) and (4.9) allow us now to calculate the correlations we are interested in terms of the currents after eliminating the fields. One obtains thus

$$P_1(\boldsymbol{k}) = Tr\{\rho_i\, J^\dagger_\mathrm{H}(\boldsymbol{k})J_\mathrm{H}(\boldsymbol{k})\} \ , \tag{4.10}$$

$$P_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = Tr\{\rho_i\, \tilde{\mathscr{T}}[J^\dagger_\mathrm{H}(\boldsymbol{k}_1)J^\dagger_\mathrm{H}(\boldsymbol{k}_2)]\mathscr{T}[J_\mathrm{H}(\boldsymbol{k}_1)J_\mathrm{H}(\boldsymbol{k}_2)]\} \ , \tag{4.11}$$

where the label H stands for the Heisenberg representation. Now the cross sections depend only on the currents and the density matrix in the initial state. The appearance of the time ordering operators $\mathscr{T}$ and $\tilde{\mathscr{T}}$ in Eqs. (4.10) and (4.11) is a reminder of the fact that the current J is here an operator.

4.2. Classical currents

Besides the fact that in this case an exact, analytical solution of the field equations is available and that the limits of this approximation are quite clear, the classical current has the important advantage that in it the space–time characteristics of the source are clearly exhibited and thus contact with approaches like the Wigner approach and hydrodynamics are made possible.

The assumption that the currents are classical implies that J is a c number and then the order in Eq. (4.11) does not matter. This approximation can be used in particle physics when the currents are produced by heavy particles (e.g. nucleons) and/or when the momentum transfer q is small compared with the momentum K of the emitting particles.[21] In Section 4.7 a new criterion for the applicability of the classical current assumption in terms of particle–antiparticle correlations will be presented.

The classical current formalism was introduced to the field of Bose–Einstein correlations in [64,65,39]. In this approach, particle sources are treated as external classical currents $J(x)$, the fluctuations of which are described by a probability distribution $P\{J\}$.

From many points of view like, e.g. understanding the space–time properties of the sources or the isotopic spin dependence of BEC this approach is superior to any other approach. This has become clear only in the last years [33,3] when a systematic investigation of the independent physical quantities which enter the dynamics of correlation functions has been made (see below).

The classical current formalism in momentum space is mathematically identical with the coherent state formalism used in quantum statistics and in particular in quantum optics (see Section 2.2), the classical currents in k-space $J(k)$ being proportional to the eigenvalues of the coherent states $|\alpha\rangle$. This explains the importance of the coherent state formalism for applications in particle and nuclear physics.

The density matrix is

$$\rho = \int \mathscr{D}J \mathscr{P}\{J\} |J\rangle\langle J| , \tag{4.12}$$

where the symbol $\mathscr{D}J$ denotes an integration over the space of functions $J(x)$, and the statistical weight $\mathscr{P}\{J\}$ is normalised to unity,

$$\int \mathscr{D}J \mathscr{P}\{J\} = 1 . \tag{4.13}$$

(The reader will recognise in Eq. (4.12) the $\mathscr{P}$-representation introduced in Section 2.2.)

Expectation values of field operators can then be expressed as averages over the corresponding functionals of the currents, e.g.,

$$\begin{aligned} &Tr(\rho a^\dagger(\boldsymbol{k}_1) \ldots a^\dagger(\boldsymbol{k}_n) a(\boldsymbol{k}_{n+1}) \ldots a(\boldsymbol{k}_{n+m})) \\ &= \prod_{j=1}^{n} \frac{(-\mathrm{i})}{\sqrt{(2\pi)^3 2E_j}} \prod_{j=m+1}^{n+m} \frac{\mathrm{i}}{\sqrt{(2\pi)^3 2E_j}} \langle J^*(\boldsymbol{k}_1) \ldots J^*(\boldsymbol{k}_n) J(\boldsymbol{k}_{n+1}) \ldots J(\boldsymbol{k}_{n+m}) \rangle . \end{aligned} \tag{4.14}$$

In the following, we shall discuss the special case where the fluctuations of the currents $J(x)$ are described by a Gaussian distribution $\mathscr{P}\{J\}$. The reasons for this choice are given in Section 2.2.

[21] For multiparticle production processes this also implies a constraint on the multiplicity and/or the momenta k of the produced particles.

As in quantum optics we write the current $J(x)$ as the sum of a chaotic and a coherent component

$$J(x) = J_{\mathrm{chaotic}}(x) + J_{\mathrm{coherent}}(x) \tag{4.15}$$

with

$$J_{\mathrm{coherent}}(x) = \langle J(x) \rangle \,, \tag{4.16}$$

$$J_{\mathrm{chaotic}}(x) = J(x) - \langle J(x) \rangle \,. \tag{4.17}$$

By definition, $\langle J_{\mathrm{chaotic}}(x) \rangle = 0$. The case $\langle J(x) \rangle \neq 0$ corresponds to *single-particle coherence*.

The Gaussian current distribution is completely determined by specifying its first two moments: the first moment coincides, because of Eq. (4.17), with the coherent component,

$$I(x) \equiv \langle J(x) \rangle \tag{4.18}$$

and the second moment is given by the 2-current correlator

$$D(x, x') \equiv \langle J(x) J(x') \rangle - \langle J(x) \rangle \langle J(x') \rangle = \langle J_{\mathrm{chaotic}}(x) J_{\mathrm{chaotic}}(x') \rangle \,. \tag{4.19}$$

4.3. Primordial correlator, correlation length and space–time distribution of the source

We come now to a more recent development [33,3] of the current formalism which has shed new light on both fundamental and applicative aspects of BEC.

There are two in principle independent aspects of physics which come together in the phenomenon of BEC in particle and nuclear physics. One refers to the *geometry* of the source and goes back to the original Hanbury-Brown and Twiss interference experiment in astronomy. The "geometry" is characterised by the size of the source, e.g. the longitudinal and transverse radius $R_{\parallel}$ and $R_{\perp}$, respectively, and the lifetime of the source R_0.

The second aspect is related to the *dynamics* of the source and is expressed through correlation lengths. In the following, we will use two correlation lengths $L_{\parallel}$, $L_{\perp}$ and a correlation time L_0.[22] As a consequence of the finite space–time size of sources in particle physics one cannot in general separate the geometry from the dynamics in the second- (and higher-)order correlation function. This separation is possible (see below) only by using simultaneously also the single inclusive cross Section [33].[23]

Assuming Gaussian currents, one may take the point of view that the purpose of measuring n-particle distributions is to obtain information about the space–time form of the coherent component, $I(x)$, and of the correlator of the chaotic components of the current, $D(x, x')$. In practice,

[22] We refer here to short range correlations. See Section 6.1.3 for a distinction between short- and long-range correlations.

[23] For systems in local equilibrium Makhlin and Sinyukov [66] introduced a length scale (called in [67] "length of homogeneity") which characterises the hydrodynamical expansion of the source and can be different from the size of the system. Further references on this topic can be found, e.g. in [68,69].

because of limited statistics it is necessary to consider simple parametrisations of these quantities, and to use the information extracted from experimental data to determine the free parameters (such as radii, correlation lengths, etc.).

The approach considered in [3] differs in some fundamental aspects from those applications of the density matrix approach in particle physics which are performed in momentum (rapidity) space and are limited usually to one dimension (see however [70] where rapidity *and* transverse momentum were considered). The approach of Ref. [3] is a *space–time* approach in which the parameters refer to the space–time characteristics of the source. This approach has important heuristic advantages compared with the momentum (rapidity) space approach as explained below.

Among other things, in the quantum statistical (QS) space–time approach the parameters of the source as defined above can be considered as effective parameters which contain already the entire information which one is interested in and which one could obtain from experiment, and thus distinguishing between directly produced particles and resonance decay products could amount to double counting. The apparent proliferation of parameters brought about by the QS approach is compensated by this heuristic and practical simplification. Furthermore, a new and essential feature of the approach of Ref. [3] as compared with previous applications of the current formalism [65,39] which assume $L = 0$, is the *finite* correlation length (time) L. This fact has important theoretical and practical consequences. It leads among other things to an effective correlation between momenta and coordinates, so that, e.g. the second-order correlation functions depend not only on the difference of momenta $q = k_1 - k_2$ but also on the sum $k_1 + k_2$. This non-stationarity property, which is observed in experiment, is usually associated with expanding sources and treated within the Wigner function formalism. However from the considerations presented above it follows that expansion is in general not a necessary condition for non-stationarity in q. It will be shown how expanding sources can be treated without the Wigner formalism, which restricts unnecessarily the applicability of the results to small q values (see Section 4.8).

The distinction between correlation lengths and radii is possible only in the current formalism; the Wigner formalism provides just a length of homogeneity.

The results which follow from the space–time approach [3] include:

(i) The existence of at least 10 independent parameters that enter into the correlation function;

(ii) new insights into the problem of partial coherence;

(iii) isospin effects: $\pi^0\pi^0$ correlations are different from $\pi^\pm\pi^\pm$ correlations; there exists a quantum statistical (anti)correlation between particles and antiparticles ($\pi^+\pi^-$ in this case). These effects are associated with the presence of squeezed states in the density matrix, which in itself is a surprising and unexpected feature in conventional strong interaction phenomenology.

It turns out that soft pions play an essential role in the experimental investigation of BEC, both with respect to the effect of particle–antiparticle correlations as well as in the investigation of the coherence of the source. Depending on the relative magnitude of the parameters of the coherent and chaotic component, soft particles can either enhance or suppress the coherence effect.

Consider first the case of an infinitely extended source. The correlation of currents at two space–time points x and y is described by a primordial correlator

$$\langle J(x)J(y)\rangle_0 = C(x - y) . \tag{4.20}$$

Note that C depends only on the difference $x-y$. The correlator $C(x-y)$ contains some characteristic length (time) scales L, the so-called correlation lengths (times).[24] In the current formalism used in [39] $L=0$.

$C(x-y)$ is a real and even function of its argument. In the rest frame of the source, it is usually parametrised by an exponential,

$$C(x-y)=C_0\exp[-|x_0-y_0|/L_0-|\mathbf{x}-\mathbf{y}|/L] \tag{4.21}$$

or by a Gaussian,

$$C(x-y)=C_0\exp[-(x_0-y_0)^2/2L_0^2-(\mathbf{x}-\mathbf{y})^2/2L^2]\ . \tag{4.22}$$

However, it should be clear that in principle any well-behaved decreasing function of $(x-y)$ is a priori acceptable, and in practice it is usually up to the experimenter to decide which particular form is more appropriate. Ansatze (4.21) and (4.22) need to be modified for the case of an expanding source (see Sections 4.8 and 4.9) where each source element is characterised not only by a correlation length L^μ but also by a four-velocity u^μ. As a matter of fact, the form of the function C is irrelevant as long as one is interested in the general statements of the theoretical quantum statistical (current) formalism. In practical applications, of course, in order to obtain concrete information about the source and the medium (i.e., about L) the form of C has to be specified. In principle, a full dynamical theory is expected to determine the functional form of the correlation function; however at present this "fundamentalist" approach is not applicable.[25] One uses instead a phenomenological approach like that reflected in Eqs. (4.21) and (4.22).

Effects of the *geometry* of the source are taken into account by introducing the space–time distributions of the chaotic and of the coherent component, $f_{ch}(x)$ and $f_c(x)$, respectively. The expectation values of the currents, $I(x)$ and $D(x,x')$, take non-zero values only in space–time regions where f_c and f_{ch} are non-zero. Thus, one may write

$$I(x)=f_c(x)\ , \tag{4.23}$$

$$D(x,x')=f_{ch}(x)C(x-x')f_{ch}(x')\ . \tag{4.24}$$

We turn now to an important new aspect of the current approach.

4.4. Production of an isospin multiplet

Following [3] we generalise the previous results to the case of an isospin multiplet and derive explicit expressions for the single inclusive distributions and correlation functions of particles that form an isotriplet (such as the π^+,π^- and π^0-mesons). For the sake of definiteness, we will refer to pions in the discussion below, but it should be understood that the formalism is applicable to an arbitrary isomultiplet.

[24] These correspond to the "coherence lengths" used in the quantum optical literature.

[25] Indeed one may hope that for strongly interacting systems lattice QCD may provide in future the correlation length L.

Consider the production of charged and neutral pions, π^+, π^- and π^0, by random currents. The initial interaction Lagrangian is written

$$\mathscr{L}_{\mathrm{int}} = J_+(x)\pi^-(x) + J_-(x)\pi^+(x) + J_0(x)\pi^0(x) . \quad (4.25)$$

The current distribution is completely characterised by its first two moments. They read for the case of an isotriplet

$$I_i(x) \equiv \langle J_i(x) \rangle, \quad i, i' = +, -, 0 ,$$
$$D_{ii'}(x, x') \equiv \langle J_i(x) J_{i'}(x') \rangle - \langle J_i(x) \rangle \langle J_{i'}(x') \rangle . \quad (4.26)$$

Invariance of the chaotic 2-current correlator under rotation in isospin space implies

$$D_{00}(x, x') = D_{+-}(x, x') \equiv D(x, x') ,$$
$$D_{++}(x, x') = D_{--}(x, x') = D_{+0}(x, x') = D_{0-}(x, x') = 0 . \quad (4.27)$$

The corresponding current distribution is

$$P\{J_i\} = (1/\mathscr{N}) \exp[-\mathscr{A}\{J_i\}] \quad (4.28)$$

where in coordinate representation

$$\mathscr{N} = \iiint \mathscr{D}J_+ \mathscr{D}J_- \mathscr{D}J_0 \exp[-\mathscr{A}\{J_i\}] , \quad (4.29)$$

and

$$\mathscr{A}\{J_i\} = \iint \mathrm{d}^4x\, \mathrm{d}^4y \Big[(J_+(x) - I_+(x)) M(x, y) (J_-(y) - I_-(y)) + \frac{1}{2}(J_0(x) - I_0(x)) M(x, y) (J_0(y) - I_0(y)) \Big] \quad (4.30)$$

with

$$M(x, x') = D^{-1}(x, x') . \quad (4.31)$$

For the general case of a partially coherent source, the single inclusive distributions of pions of charge i ($i = +, -, 0$) can be expressed as the sum of a chaotic component and a coherent component,

$$(1/\sigma)\mathrm{d}\sigma^i/\mathrm{d}\omega = (1/\sigma)\mathrm{d}\sigma^i/\mathrm{d}\omega|_{\mathrm{chaotic}} + (1/\sigma)\mathrm{d}\sigma^i/\mathrm{d}\omega|_{\mathrm{coherent}} \quad (4.32)$$

with

$$(1/\sigma)\mathrm{d}\sigma^i/\mathrm{d}\omega|_{\mathrm{chaotic}} = D(k, k) , \quad (4.33)$$

and

$$(1/\sigma)\mathrm{d}\sigma^i/\mathrm{d}\omega|_{\mathrm{coherent}} = |I(k)|^2 . \quad (4.34)$$

In general, the chaoticity parameter p will be momentum dependent,

$$p(k) = D(k,k)/(D(k,k) + |I(k)|^2) . \tag{4.35}$$

To write down the correlation functions in a concise form, one introduces the normalised current correlators

$$d_{rs} = D(k_r,k_s)/[D(k_r,k_r)\cdot D(k_s,k_s)]^{1/2}, \qquad \tilde{d}_{rs} = D(k_r,-k_s)/[D(k_r,k_r)\cdot D(k_s,k_s)]^{1/2} , \tag{4.36}$$

where the indices r,s label the particles. Note that $\tilde{d}$ is the *same* function as d but for the change of sign of one of its variables. One may express the correlation functions in terms of the magnitudes and the phases of the correlators:

$$\begin{aligned} &T_{rs} \equiv T(k_r,k_s) = |d(k_r,k_s)|, \qquad \tilde{T}_{rs} \equiv \tilde{T}(k_r,k_s) = |\tilde{d}(k_r,k_s)| , \\ &\phi^{\mathrm{ch}}_{rs} \equiv \phi^{\mathrm{ch}}(k_r,k_s) = \operatorname{Arg} d(k_r,k_s), \qquad \tilde{\phi}^{\mathrm{ch}}_{rs} \equiv \tilde{\phi}^{\mathrm{ch}}(k_r,k_s) = \operatorname{Arg} \tilde{d}(k_r,k_s) \end{aligned} \tag{4.37}$$

and the phase of the coherent component

$$\phi^{\mathrm{c}}_r \equiv \phi^{\mathrm{c}}(k_r) = \operatorname{Arg} I(k_r) . \tag{4.38}$$

The same notation will be used for the chaoticity parameter

$$p_r \equiv p(k_r) . \tag{4.39}$$

The two-particle correlation function is

$$C_2^{+\,+}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + 2\sqrt{p_1(1-p_1)\cdot p_2(1-p_2)}\,T_{12}\cos(\phi^{\mathrm{ch}}_{12} - \phi^{\mathrm{c}}_1 + \phi^{\mathrm{c}}_2) + p_1p_2T_{12}^2 . \tag{4.40}$$

In [3] higher-order correlation functions up to and including order 5 are given.

In the absence of single-particle coherence the two-particle correlation functions for different pairs of π^+,π^-,π^0 mesons read

$$\begin{aligned} &C_2^{+\,+}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + |d_{12}|^2, \qquad C_2^{+\,-}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + |\tilde{d}_{12}|^2 , \\ &C_2^{+\,0}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1, \qquad C_2^{00}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + |d_{12}|^2 + |\tilde{d}_{12}|^2 \end{aligned} \tag{4.41}$$

These results [71] were surprising in that they disagreed with some of the preconceived notions on Bose–Einstein correlations. For instance, it was commonly assumed that without taking into account final state interactions and in the absence of coherence, the maximum of the two-particle correlation of identical pions is 2 (for $\boldsymbol{k}_1 = \boldsymbol{k}_2$). It was also assumed that there are no correlation effects among different kinds of pions because these particles are not identical. (This last assumption is even sometimes used in normalising the experimental data on $C_2^{\pm\,\pm}$ with respect to $C_2^{+\,-}$.)

Results (4.41) show that these assertions are not necessarily true. In particular, looking at the two pion correlations one can see that in addition to the familiar correlations of identical particles (the terms $|d_{12}|^2$) there are particle–antiparticle – in this case, $\pi^+\pi^-$ – correlations (the terms $|\tilde{d}_{12}|^2$). The π^0 has both terms, as it is identical with its antiparticle. Essentially, this last fact is the explanation for the appearance of the "surprising" effects. Obviously it is a specific quantum field effect. It will be shown in Section 4.8, that for soft pions and for small lifetimes of the source the terms $|\tilde{d}_{12}|^2$ can in principle become comparable with the conventional terms $|d_{12}|^2$. This implies that the distribution $C_2^{00}(\boldsymbol{k}_1, \boldsymbol{k}_2)$ of two neutral pions can be as large as 3, and the maximum value of C_2^{+-} is 2 (instead of 1). The corresponding limit of C_3^{000} is 15 (instead of 6), and that of C_3^{++-} is 6 (instead of 2). However, it should be noted that these are merely upper limits, which for massive particles are not reached except for sources of infinitesimally small lifetimes. For soft photons however the situation is different (see below).

The "new" terms, proportional to $\langle a_\ell(k_1)a_\ell(k_2)\rangle$ are due to the non-stationarity (in k space) of the source. While in quantum optics time stationarity is the rule, in particle physics this is not the case because of the finite lifetime and finite radius of the sources. The existence of a non-vanishing expectation value of the products $a(k_1)a(k_2)$ is what one would expect (see Section 2.2) from two-particle coherence (squeezing), just as $\langle a(k)\rangle \neq 0$ follows from ordinary (one-particle) coherence (note that the latter has not been assumed here). The fact that squeezing which, as mentioned above, is quite an exceptional situation in optics, discovered only recently, is a natural consequence of the formalism for present particle physics, is possibly one of the most startling results obtained recently in BEC. A characteristic feature of isospin squeezed states is that they are two-mode states and that for static sources they lead to anti-correlations. Finally, it should be pointed out again that the above results – in particular, the fact that C_2^{00} in general differs from C_2^{--} – are consistent with isospin symmetry.

In closing this section one should note that the existence of particle–antiparticle correlations is not restricted to pions but applies also to other systems, e.g. neutral kaons. In principle, thus there exist also $K_0\bar{K}_0$ quantum statistical correlations. However, since K_0 particles cannot be observed except in linear combinations with $\bar{K}_0$ in the form of K_s and K_l, the QS particle–antiparticle correlation effect has to be disentangled from the K_sK_s or K_lK_l Bose–Einstein correlation (K_s and K_l are of course bosons and thus subject to BEC) which exist also in the wave-function formalism which ignores the intrinsic "new" $K_0\bar{K}_0$ correlation. As a matter of fact K_sK_s correlations have been observed experimentally; however, no attempt has been made so far to extract from them the surprising effects.[26] (For more recent experiments see [72] and for a theoretical analysis of the "old" effects and their possible application in CP violation phenomena see [73].)[27]

[26] Because of their larger mass as compared with that of pions, these effects may be even more quenched than in the case of pions, except for sources of very short lifetime (see Section 4.8).

[27] The analysis in the preceding subsections referred to the production of an isotriplet assuming just the symmetry between the isospin components (see Eq. (4.25)) of the current. In principle, for strong interactions the conservation of isospin I must also be considered. While the chaotic part is not affected by this condition, the coherent component is influenced by conservation of isospin [74]. In particular, this can lead to an additive positive term in the correlation function and thus to an increase of the bounds of BEC for pions. It remains to be seen whether this effect can be distinguished from the effect of long-range correlations (see Section 6.1.3). Moreover in hadronic reactions and in particular those involving nuclei such an effect would be suppressed because of the following circumstance. The initial state has to be averaged over all components of isospin I which is a first "diluting" factor. Furthermore, the effect is appreciable only for low total isospin. This total isospin has to be shared by the chaotic component I_{ch} and the coherent component I_c: $I = I_c + I_{ch}$. The first one arises mostly from resonances with different isospin values, so that even if the total isospin takes its minimum value ($I = 0$), I_{ch} and therefore I_c can take larger values.

4.4.1. An illustrative model of uncorrelated point-like random sources

To clarify the origin of different terms in the functions $C_2^{ab}(k_1, k_2)$, let us consider, a source consisting of N point-like random sources

$$J_a(x) = \sum_{i=1}^{N} j_a(x_i)\delta(t - t_i)\delta^3(\boldsymbol{x} - \boldsymbol{x}_i), \quad a = +, -, 0 \tag{4.42}$$

and assume that the currents $j_a(x_i)$ at different points x_i are mutually independent and have the same statistical properties, i.e.

$$\langle j_a^*(x_i) j_b(x_j) \rangle = \delta_{ij} \langle j_a^* j_b \rangle . \tag{4.43}$$

We also assume in this section that

$$\langle j_a \rangle = 0 , \tag{4.44}$$

i.e. ignore a possible coherent component $\langle j_a \rangle$ to make the presentation more transparent.

Now the one-particle distribution is

$$\langle J_a^*(\boldsymbol{k}) J_a(\boldsymbol{k}) \rangle = N \cdot \langle j_a^* j_a \rangle \tag{4.45}$$

and the two-particle distribution takes the form

$$\begin{aligned}
&\langle J_a^*(k_1)\ J_a(k_1)\ J_b^*(k_2)\ J_b(k_2) \rangle \\
&\quad = \sum_{i=1}^{N} \langle j_a^*(x_i) j_a(x_i) j_b^*(x_i) j_b(x_i) \rangle + \sum_{i \neq j}^{N} \langle j_a^*(x_i) j_a(x_i) \rangle \langle j_b^*(x_j) j_b(x_j) \rangle \\
&\qquad + \sum_{i \neq j}^{N} \langle j_b^*(x_i) j_a(x_i) \rangle \langle j_a^*(x_j) j_b(x_j) \rangle e^{i(k_1 - k_2)(x_i - x_j)} \\
&\qquad + \sum_{i \neq j}^{N} \langle j_a^*(x_j) j_b^*(x_j) \rangle \langle j_a(x_i) j_b(x_i) \rangle e^{i(k_1 + k_2)(x_i - x_j)} .
\end{aligned} \tag{4.46}$$

Let us consider separately the four different terms on the right-hand side of Eq. (4.46). The first term corresponds to two particles being emitted from a single point (Fig. 6a); it is proportional to the number of emitting points.

The second term describes an independent emission of two particles from different points (Fig. 6b).

The third term, being non-zero for $a = b$, describes an interference effect of direct and exchange diagrams, characteristic of identical particles, emitted from different points (Fig. 6c). This is the usual BE-correlation effect.

The fourth term describes an interference of two-particle emissions from different points (Fig. 6d) ("two-particle sources"). It is non-zero for real currents with $a = b$ ($\pi^0\pi^0$) and for complex currents with $J_a^* = J_b$ ($\pi^+\pi^-$), that is for particle–antiparticle associative emission. In this simple model, it represents the "surprising" effects discussed above.

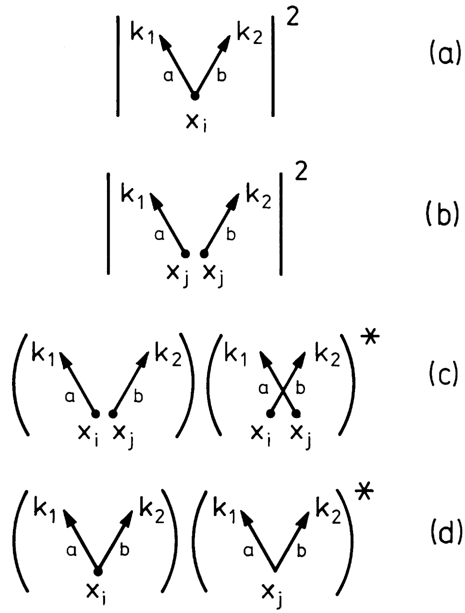


Fig. 6. Diagrams contributing to different terms of the two-particle correlator in the point-like random source model (from Ref. [3]).

This diagrammatic illustration of the "surprising" BE-correlations is due to Bowler [75], who found that these effects derived for the first time in Ref. [71] can be understood in terms of the qualitative considerations mentioned above. Bowler derived the "new" effects from the string model.

4.5. Photon interferometry. Upper bounds of BEC

The advantage of photon BEC resides in the fact that photons are not influenced by final state interactions. Photons present also an interesting subject of theoretical research from the general BEC point of view, since they are spin-one bosons while pions and kaons used in hadronic BEC are scalar particles. We shall see below that this supplementary degree of freedom has specific implications for BEC. Last but not least, photon correlations are for various reasons, discussed below, of particular interest in the search of quark matter. We present some of the results of [76], which contain as a special case those of [77] and where these topics are discussed.

Consider a heavy ion reaction where photons are produced through bremsstrahlung from protons in independent proton–neutron collisions.[28] The corresponding elementary dipole currents are

$$j^{\lambda}(k) = (\mathrm{i}e/mk^0)\boldsymbol{p}\cdot\boldsymbol{\varepsilon}_{\lambda}(k)\ , \tag{4.47}$$

[28] Photon emission from proton–proton collisions is suppressed because it is of quadrupole form.

where $\boldsymbol{p} = \boldsymbol{p}_{\rm i} - \boldsymbol{p}_{\rm f}$ is the difference between the initial and the final momenta of the proton, $\boldsymbol{\varepsilon}_\lambda$ is the vector of linear polarization and k the photon momentum; e and m are the charge and mass of the proton, respectively. The total current is written as

$$J^\lambda(k) = \sum_{n=1}^{N} e^{ikx_n} j_n^\lambda(k) . \tag{4.48}$$

For simplicity, we will discuss in the following only the case of pure chaotic currents $\langle J^\lambda(k)\rangle = 0$ and refer for coherence effects to the original literature [76,77]. In analogy to the considerations of the previous subsection the index n labels the independent nucleon collisions which take place at different space–time points x_n. These points are assumed to be randomly distributed in the space–time volume of the source with a distribution function $f(x)$ for each elementary collision. The current correlator is proportional to products of the form

$$\langle J^{\lambda_1}(k_1) J^{\lambda_2}(-k_2)\rangle = \varepsilon^i_{\lambda_1}(k_1)\left(\sum_{n=1}^{N} \langle \boldsymbol{p}^i_n \boldsymbol{p}^j_n\rangle\right)\varepsilon^j_{\lambda_2}(k_2) . \tag{4.49}$$

For central collisions due to the axial symmetry around the beam direction one has for the momenta the tensor decomposition

$$\langle \boldsymbol{p}^i_n \boldsymbol{p}^j_n\rangle = \tfrac{1}{3}\sigma_n \delta^{ij} + \delta_n l^i l^j , \tag{4.50}$$

where l is the unit vector in the beam direction and σ_n, δ_n are real positive constants. In [77] an isotropic distribution of the momenta was assumed. This corresponds to the particular case $\delta_n = 0$. The generalisation to the form (4.50) is due to [76]. The summation over polarisation indexes is performed using the relations

$$\langle(\varepsilon^i \cdot \boldsymbol{p}_l)(\varepsilon^j \cdot \boldsymbol{p}_{l'})\rangle = \tfrac{1}{3}(\varepsilon^i \cdot \varepsilon^j)\delta_{ll'} , \tag{4.51}$$

$$\sum_{\lambda=1}^{2} \varepsilon^i_\lambda(k)\varepsilon^j_\lambda(k) = \delta^{ij} - \boldsymbol{n}^i \boldsymbol{n}^j , \tag{4.52}$$

where $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}|$.

We write below the results for the second-order correlation function

$$C_2(k_1, k_2) = \rho_2(k_1, k_2)/\rho_1(k_1)\rho_1(k_2) \tag{4.53}$$

for two extreme cases:

(1) Uncorrelated elementary currents (isotropy) ($\sigma \gg \delta$)

$$C_2(k_1, k_2;\ \sigma \neq 0,\ \delta = 0) = 1 + \tfrac{1}{4}[1 + (\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)^2][|\tilde{f}(k_1 - k_2)|^2 + |\tilde{f}(k_1 + k_2)|^2] , \tag{4.54}$$

leading to an intercept

$$C_2(k, k) = \tfrac{3}{2} + \tfrac{1}{2}|\tilde{f}(2k)|^2 \tag{4.55}$$

limited by values $(\tfrac{3}{2}, 2)$.

(2) Strong anisotropy ($\sigma \ll \delta$)

$$C_2(k_1, k_2;\ \sigma = 0,\ \delta \neq 0) = 1 + |\tilde{f}(k_1 - k_2)|^2 + |\tilde{f}(k_1 + k_2)|^2 \tag{4.56}$$

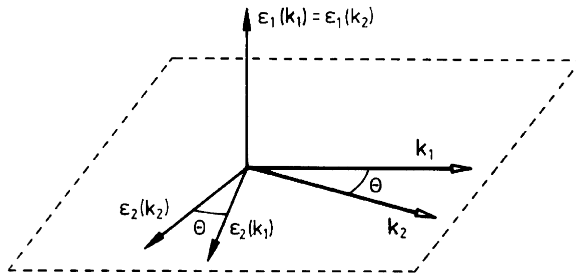


Fig. 7. The linear polarisation vectors $\varepsilon_\lambda(k)$ for two photons with momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ (from Ref. [82]).

with an intercept

$$C_2(k,k) = 2 + |\tilde{f}(2k)|^2 \tag{4.57}$$

limited this time by values (2,3). Note that due to the form of the photon–current interaction (4.47) in this (strong anisotropy) case the photons emerge practically completely polarised so that the summation over polarisations does not affect the correlation.

These results are remarkable among other things because they illustrate the specific effects of photon spin on BEC. Thus while for (pseudo)scalar pions the intercept is a constant (2 for charged pions and 3 for neutral ones) even for unpolarised photons the intercept is a function of k.

For a graphical illustration and explanation of this fact see Fig. 7. It is seen that to perform the summation over polarisation implied by Eq. (4.51) only one direction of the linear polarisation can be chosen to be equal for both photons, while the other polarisation direction differs by the angle θ between the momenta $\boldsymbol{k}_1, \boldsymbol{k}_2$.

One thus finds that, while for a system of charged pions (i.e. a mixture of 50% positive and 50% negative) the maximum value of this intercept $Max\, C_2(k,k)$ is 1.5, for photons $Max\, C_2(k,k)$ exceeds this value and this excess reflects the space–time properties (represented by $\tilde{f}(k)$), the degree of (an)isotropy of the source represented by the quantities σ and δ, and the supplementary degree of freedom represented by the photon spin.

As a consequence of the fact that $\tilde{f}$ is a decreasing function of its argument, in Eqs. (4.54) and (4.56) the terms with $\tilde{f}(k_1 + k_2)$ are in general smaller than the terms with $\tilde{f}(k_1 - k_2)$, except for small momenta k.

The fact that the differences between charged pions and photons are enhanced for soft photons reminds us of a similar effect found with neutral pions (see Section 4.4). Neutral pions are in general more bunched than identically charged ones and this difference is more pronounced for soft pions. This similarity is not accidental, because photons as well as π^0 particles are neutral and this circumstance has quantum field theoretical implications which are also mentioned below.

We see thus that photon BEC can provide information both about the space–time form of the source represented by f and the dynamics which are represented by δ.[29,30]

[29] See [78,79] where photon correlation experiments in low energy (100 MeV/nucleon) heavy ion reactions are reported. For a theoretical discussion of these experiments see [80].

[30] The relation between photon interferometry and the formation length of photons is discussed in [81].

The results on photon correlations presented above refer to the case where the sources are "static" i.e. not expanding. Expanding sources were considered in [82] within a covariant formalism.

Some of the results above, in particular Eqs. (4.54) and (4.55), which had been initially derived by Neuhauser [77], were challenged by Slotta and Heinz [83]. Among other things, these authors claim that for photon correlations due to a chaotic source "the only change relative to 2-pion interferometry is a statistical factor $\frac{1}{2}$ for the overall strength of the correlation which results from the experimental averaging over the photon spin". In [83] an intercept $\frac{3}{2}$ is derived which is in contradiction with the results presented above and in particular with Eq. (4.55) where besides the factor $\frac{3}{2}$ there appears also the k dependent function $\frac{1}{2}|\tilde{f}(2k)|^2$. Similar statements can be found in previous papers [84–87] where more detailed applications concerning heavy ion reactions based on this assertion of [83] are presented.

Some of the papers quoted above were criticised immediately after their publication in [88,89,82] and the paper [83] was written with the intention to settle this "controversy".

It should be pointed out here that the reason for the difference between the results of [77,76] on the one hand and those of Ref. [83] on the other is mainly that in [83] a formalism was used which is less general than that used in [77,76] and which is inadequate for the present problem. This implies among other things that unpolarised photons cannot be treated in the way proposed in [83] and that the results of [77,76] are correct.

In [83] the following formula for the second-order correlation function is used:

$$C(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \frac{\tilde{g}_{\mu\nu}(\boldsymbol{q},\boldsymbol{K})\tilde{g}^{\nu\mu}(-\boldsymbol{q},\boldsymbol{K})}{\tilde{g}^{\mu}_{\mu}(\boldsymbol{0},\boldsymbol{k}_1)\tilde{g}^{\mu}_{\mu}(\boldsymbol{0},\boldsymbol{k}_2)} \,. \tag{4.58}$$

Here $\tilde{g}$ is the Fourier transform of a source function, $\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_2$ and $\boldsymbol{K} = \frac{1}{2}(\boldsymbol{k}_1 + \boldsymbol{k}_2)$. This formula is a particular case of a more general formula for the second-order correlation function derived by Shuryak [64] using a model of uncorrelated sources, when emission of particles from the same space–time point is negligible (see Section 4.4.1).

Since this equation is sometimes used in the recent literature without giving the reader the possibility of evaluating the approximations used in its derivation, we will sketch this derivation in the following.

In [64] one starts with the current correlator

$$\langle J_i^*(x_1)J_j(x_2)\rangle = \delta_{ij}J_i(x,\Delta x) \,, \tag{4.59}$$

where $J_i(x)$ is the current emitted by point x and

$$x = (x_1 + x_2)/2, \qquad \Delta x = x_1 - x_2 \,. \tag{4.60}$$

Eq. (4.59) assumes that the individual currents ($i \neq j$) are uncorrelated. With the notation $\tilde{I}_i(q,K)$ for the Fourier transform of $I_i(x,\Delta x)$ and $\tilde{I}(q,K) \equiv \sum_i \tilde{I}(q,K)$ the inclusive single-particle distribution reads

$$W(k) = \left\langle \left| \sum_j \int e^{ikx} J_j(x)\,d^4x \right| \right\rangle = \sum_i \tilde{I}_i(0,k) = \tilde{I}(0,k) \tag{4.61}$$

and the two-particle distribution is given by

$$W(k_1 k_2) = \left\langle \left| \sum_{i,j} \int (e^{ik_1x_1 + ik_2x_2} + e^{ik_1x_2 + ik_2x_1}) J_i(x_1)J_j(x_2)\,dx_1\,dx_2 \right|^2 \right\rangle \tag{4.62}$$

or finally

$$W(k_1,k_2) = \tilde{I}(0,k_1)\tilde{I}(0,k_2) + |\tilde{I}(q,K)|^2 + \sum_i [\langle J_i^*(k_1)J_i^*(k_2)J_i(k_1)J_i(k_2)\rangle - \tilde{I}_i(0,k_1)\tilde{I}_i(0,k_2) - |\tilde{I}_i(q,K)|^2] \,. \tag{4.63}$$

The first term on the r.h.s. of Eq. (4.63) is the product of one-particle distributions and the second term is the conventional interference term, corresponding to Fig. 6c. By going over to the Wigner source function (see also Section 4.9)

$$g_{\mu\nu}(x,K)=\int \mathrm{d}^4y\,\mathrm{e}^{-\mathrm{i}Ky}\langle J^*_\mu(x+\tfrac{1}{2}y)J_\nu(x-\tfrac{1}{2}y)\rangle\ , \tag{4.64}$$

these first two terms result in Eq. (4.58) of [83]. However as is clear from Eq. (4.63) there exists also a third term, neglected in Eq. (4.58) and which corresponds to the simultaneous emission of two particles from a single point (x_i) as indicated in Fig. 6d. While for massive particles this term is in general suppressed, this is not true for massless particles and in particular for soft photons. In [77,76] this additional term had not been neglected as it was done in [83] and therefore it is not surprising that Ref. [83] could not recover the results of Refs. [77,76]. The neglect of the term corresponding to emission of two particles from the same space–time point is not permitted in the present case. As mentioned in Section 4.4.1, in a model of uncorrelated point-like random sources like the present one, emission of particles from the same space–time point corresponds in a first approximation to particle–antiparticle correlations and this type of effect leads also to the difference between BEC for identically charged pions and the BEC for neutral pions. This is so because neutral particles coincide with the corresponding antiparticles. (As a consequence of this, e.g. while for charged pions the maximum of the intercept is 2, for neutral pions it is 3 (see Section 4.4).) Photons being neutral particles, similar effects like those observed for π^0–s are expected and indeed found (see above).

This inconsequent application of the current formalism invalidates the conclusions of Ref. [83] and confirms and strengthens the criticism expressed in [88,89,82] of the papers [84–87]. The fact that for unpolarised photons $Max\,C_2(k,k)$ is 2, can be understood by realising that a system of unpolarised photons consists on the average of 50% photons with the same helicities and 50% photons with opposite helicities. The first ones contribute to the maximum intercept (of the unpolarized system) with a factor of 3 and the last ones with a factor of 1 (corresponding to unidentical particles).

For the sake of clarification it must be mentioned that Ref. [83] contains also other incorrect statements. Thus the claim in [83] that the approach by Neuhauser "does not correctly take into account the constraints from current conservation" is unfounded as can be seen from Eq. (4.51) which is an obvious consequence of current conservation (see e.g. Eq. (7.61) in [90]). Last but not least the statement that because the tensor structure in Eq. (20) of Ref. [82] is parametrised in terms of k_1 and k_2 separately "instead of only in terms of K, leading to spurious terms in the tensor structure which eventually result in their spurious momentum-dependent prefactor" has also to be qualified. As mentioned above, Eq. (4.58) to which this observation about the K dependence of [83] refers is not general enough for the problem of photon interferometry.

4.6. Coherence and lower bounds of Bose–Einstein correlations

We mentioned in the previous subsections that the intercepts of the second- and higher-order correlation functions can deviate from the canonical values derived within the wave function formalism. This effect is important for at least three reasons:

(i) It illustrates the limitations of the wave-function approach.

(ii) It can in principle (provided other effects like final state interactions are taken into account) be used for the determination of the degree of coherence.

(iii) It can serve as a test of models of BEC, since the value of the intercept follows from very general quantum statistical considerations, in particular the Gaussian nature of the density matrix.

In most BEC models the intercept is identical to the maximum of the correlation function and therefore it can be studied by limiting the discussion to chaotic sources as was done in the previous subsection where the upper bounds of correlation functions were investigated. On the other hand, the minimum of the correlation functions is determined both by the form of the density matrix and the amount of coherence (see Ref. [88]) because coherence leads to a decrease of the correlation function. This will be illustrated below by discussing the lower bounds of this function. We will show among other things (see [3]) that in the quite general case of a Gaussian density matrix, for a purely chaotic system the two-particle correlation function must always be greater than one. On the other hand, in the presence of a coherent component the correlation function may take values below unity. Some implications for experimental and theoretical results found in the literature will be discussed here as well as in Section 5.1.6.

We have seen in Section 4.4 that for identically charged bosons (e.g., π^+) the two-particle correlation function reads

$$C_2^{++}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + 2\sqrt{p_1(1-p_1)\cdot p_2(1-p_2)}\,T_{12}\cos(\phi_{12}^{\mathrm{ch}} - \phi_1^{\mathrm{c}} + \phi_2^{\mathrm{c}}) + p_1 p_2 T_{12}^2 \ . \tag{4.65}$$

For neutral bosons like photons, or π^0's, the terms $\tilde{d}(k_r,k_s)$ also appear (see Eq. (4.36)) in the BEC function:

$$C_2^{00}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + 2\sqrt{p_1(1-p_1)\cdot p_2(1-p_2)}\,T_{12}\cos(\phi_{12}^{\mathrm{ch}} - \phi_1^{\mathrm{c}} + \phi_2^{\mathrm{c}}) + p_1 p_2 T_{12}^2$$
$$+ 2\sqrt{p_1(1-p_1)\cdot p_2(1-p_2)}\,\tilde{T}_{12}\cos(\tilde{\phi}_{12}^{\mathrm{ch}} - \phi_1^{\mathrm{c}} - \phi_2^{\mathrm{c}}) + p_1 p_2 \tilde{T}_{12}^2 \ . \tag{4.66}$$

Let us first consider the case of a purely chaotic source. Insertion of $p(k) \equiv 1$ in Eqs. (4.65) and (4.66) immediately yields $C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) \geq 1$. In the case of partial coherence, the terms containing cosines come into play and consequently C_2 may take values below unity. Eqs. (4.65) and (4.66) imply that $C_2^{--}(\boldsymbol{k}_1,\boldsymbol{k}_2) \geq 2/3$ and $C_2^{00}(\boldsymbol{k}_1,\boldsymbol{k}_2) \geq 1/3$. Because of the cosine functions in (4.65) and (4.66) one would expect C_2 as a function of the momentum difference q to oscillate between values above and below 1.

Such a behaviour of the Bose–Einstein correlation function has been observed in high-energy e^+–e^- collision experiments (see e.g., Ref. [91]), but apparently not in hadronic reactions. This observation was interpreted as a consequence of final state interactions in Ref. [75]. If final state interactions determine this effect, it is unclear why the effect is not seen in hadronic reactions. On the other hand, if coherence is responsible for it, this would be easier to understand. Indeed multiplicity distributions of secondaries in e^+e^- reactions are much narrower (almost Poisson-like) than in pp reactions, which is consistent with the statement that hadronic reactions are more chaotic than e^+–e^- reactions [92].[31]

So far, two methods have been proposed for the detection of coherence in BEC: the intercept criterion [93] ($C_2(\boldsymbol{k},\boldsymbol{k}) < 2$) and the two-exponent structure of C_2 [28]. Both these methods have their difficulties because of statistics problems or other effects. The observation of $C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) < 1$ could constitute a third criterion for coherence.

In [84,85] the two-particle correlation function has been calculated for photons emitted from a longitudinally expanding system of hot and dense hadronic matter created in ultrarelativistic nuclear collisions. For such a system, the particles are emitted from a large number of independent source elements (fluid elements), and consequently one would expect the multiparticle final state to be described by a Gaussian density matrix. However, although the system is assumed to be purely chaotic the correlation function calculated in [84] is found to take values significantly below unity.

[31] This statement is not necessarily in contradiction with the empirical observation that the λ factor in e^+–e^- reactions appears in general to be larger than in p–p reactions, given the fact that λ is not a true measure of coherence.

Clearly, this is in contradiction with the general result derived above from quantum statistics ($C_2 \geq 1$ for a chaotic system).

The reason for this violation of the general bounds derived for a purely chaotic source is in this concrete case the use of an inadequate approximation in the evaluation of the space–time integrals. However as pointed out in [89] the expression for the two-particle inclusive distribution used in Ref. [84] (Eq. (3) of that paper), which in our notation takes the form

$$P_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = \int d^4x_1 \int d^4x_2 g(x_1, k_1) g(x_2, k_2)[1 + \cos((k_1 - k_2)(x_1 - x_2))] , \tag{4.67}$$

is also unsatisfactory[32] because for certain physical situations it can lead to values below unity for the two-particle correlation function even if the integrations are performed exactly. To see this, consider, e.g., the simple ansatz

$$g(x, k) = const. \exp[-\alpha(\boldsymbol{x} - \beta\boldsymbol{k})^2]\delta(t - t_0) , \tag{4.68}$$

where α and β are free parameters. The expression for $P_2(\boldsymbol{k}_1, \boldsymbol{k}_2)$ used in Ref. [84] then yields

$$C_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = 1 + \exp[-\boldsymbol{q}^2/2\alpha]\cos[\beta\boldsymbol{q}^2] . \tag{4.69}$$

Clearly, if β exceeds α^{-1} the above expression will oscillate and take values below unity. On the other hand, in the current formalism (see below) one obtains with the same ansatz for g

$$C_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = 1 + \exp[-\boldsymbol{q}^2/2\alpha] \geq 1 . \tag{4.70}$$

Thus, Eq. (4.67) can lead to values $C_2 < 1$ if there is a strong correlation between the momentum of a particle and the space–time coordinate of the source element from which it is emitted. Such correlations between x and k can occur in the case of an expanding source. The reason for this pathological behaviour is that the simultaneous specification of coordinates and momentum as implied by Eq. (4.67) is constrained in quantum mechanics by the Heisenberg uncertainty relation and any violation of this constraint leads necessarily to a violation of quantum mechanics. This violation manifests itself sometimes, as in the present case, through a violation of the conservation of probability. This phenomenon is also met when using the Wigner function, which for this reason cannot always be associated with a bonafide probability amplitude. We will discuss this problem also in Section 5.1.6.

The above considerations concerning bounds for the BEC functions refer to the case of a Gaussian density matrix. In general, a different form of the density matrix may yield correlation functions that are not constrained by the bounds derived here. For instance, we have seen in Section 2.2 that for squeezed states C_2 can take arbitrary positive values. Moreover, for particles produced in high-energy hadronic or nuclear collisions, the fluctuations of quantities such as impact parameter or inelasticity may introduce additional correlations which may also affect the bounds of the BEC functions.

[32] This formula appears apparently for the first time in [94] and was criticised (for other reasons) already in [95]. It is nevertheless used in certain event generators for heavy ion reactions (see Section 4.10).

4.7. Quantum currents

The results derived in the previous section, in particular the isospin dependence of BEC, were obtained in the assumption that the currents were classical. The question arises up to what point these conclusions survive in a fully quantum treatment of the problem. It would also be important to get a more precise criterion for the phenomenological applicability of the classical assumption, besides the no-recoil prescription. This question was discussed in [96] where it was found that the "surprising" effects not only persist when the currents are quantum, but that they can serve as an experimental estimate of the size of the quantum corrections. We shall sketch briefly in the following the results of Ref. [96].

As in the classical current case one starts with the interaction Lagrangian

$$L_{\mathrm{int}}(x) \equiv J_{(+)}(x)\pi^{(-)}(x) + J_{(-)}(x)\pi^{(+)}(x) + J_0(x)\pi^0(x) \tag{4.71}$$

The currents $J^{(+)}, J^{(-)}, J^0$ are *operators* which we assume again for simplicity as not depending on the $\pi^{(\pm)}$ and π^0 fields ($[J,\pi]=0$). Taking into account the different isospin components in Eqs. (4.10) and (4.11) we find that as in the classical current case the single and double inclusive cross sections depend on these components, e.g.[33]

$$G_1^{(-)}(\boldsymbol{k}) = Tr\{\rho_i J_{\mathrm{H}}^{(+)}(-k)J_{\mathrm{H}}^{(-)}(k)\}\ , \tag{4.72}$$

$$G_2^{(-+)}(\boldsymbol{k}_1,\boldsymbol{k}_2) = Tr\{\rho_i \tilde{\mathscr{T}}[J_{\mathrm{H}}^{(+)}(-k_1)J_{\mathrm{H}}^{(-)}(-k_2)]\mathscr{T}[J_{\mathrm{H}}^{(-)}(k_1)J_{\mathrm{H}}^{(+)}(k_2)]\}\ . \tag{4.73}$$

From now on we shall omit the label H and assume that all operators are written in the Heisenberg representation. Assuming a Gaussian density matrix one gets

$$G_1^{(0)}(\boldsymbol{k}) = F^n(k,k)\ , \tag{4.74}$$

$$G_1^{(-)}(\boldsymbol{k}) = G_1^{(+)}(\boldsymbol{k}) = F^{\mathrm{ch}}(k,k)\ , \tag{4.75}$$

$$G_2^{(--)}(\boldsymbol{k}_1,\boldsymbol{k}_2) = G_1^{(-)}(\boldsymbol{k}_1)G_1^{(-)}(\boldsymbol{k}_2) + |F^{\mathrm{ch}}(k_1,k_2)|^2\ , \tag{4.76}$$

$$G_2^{(00)}(\boldsymbol{k}_1,\boldsymbol{k}_2) = G_1^{(0)}(\boldsymbol{k}_1)G_1^{(0)}(\boldsymbol{k}_2) + |F^n(k_1,k_2)|^2 + |\Phi^{\mathrm{n}}(k_1,k_2)|^2\ , \tag{4.77}$$

$$G_2^{(-+)}(\boldsymbol{k}_1,\boldsymbol{k}_2) = G_1^{(-)}(\boldsymbol{k}_1)G_1^{(+)}(\boldsymbol{k}_2) + |\Phi^{\mathrm{ch}}(-k_1,k_2)|^2\ , \tag{4.78}$$

where functions F and Φ are defined for charged particles (upper index *ch*) and for neutral ones (upper index *n*) as follows:

$$F^{\mathrm{ch}}(k_1,k_2) \equiv \langle\!\langle \mathscr{T}_{\mathrm{c}}\{J_{\oplus}^{(+)}(-k_1)J_{\ominus}^{(-)}(k_2)\}\rangle\!\rangle = \langle\!\langle J^{(+)}(-k_1)J^{(-)}(k_2)\rangle\!\rangle\ , \tag{4.79}$$

$$\Phi^{\mathrm{ch}}(k_1,k_2) \equiv \langle\!\langle \mathscr{T}_{\mathrm{c}}\{J_{\oplus}^{(+)}(-k_1)J_{\oplus}^{(-)}(k_2)\}\rangle\!\rangle = \langle\!\langle \tilde{\mathscr{T}}\{J^{(+)}(-k_1)J^{(-)}(k_2)\}\rangle\!\rangle\ , \tag{4.80}$$

$$F^{\mathrm{n}}(k_1,k_2) \equiv \langle\!\langle \mathscr{T}_{\mathrm{c}}\{J_{\oplus}^{(0)}(-k_1)J_{\ominus}^{(0)}(k_2)\}\rangle\!\rangle = \langle\!\langle J^{(0)}(-k_1)J^{(0)}(k_2)\rangle\!\rangle\ , \tag{4.81}$$

$$\Phi^{\mathrm{n}}(k_1,k_2) \equiv \langle\!\langle \mathscr{T}_{\mathrm{c}}\{J_{\oplus}^{(0)}(-k_1)J_{\oplus}^{(0)}(k_2)\}\rangle\!\rangle = \langle\!\langle \tilde{\mathscr{T}}\{J^{(0)}(-k_1)J^{(0)}(k_2)\}\rangle\!\rangle\ . \tag{4.82}$$

[33] For reasons of notational simplicity we have replaced $J^{\dagger}(k)$ by $J(-k)$.

In contrast to the classical current approach [3] which deals with only one type of two-current correlator we have here two different kinds of two-current correlators depending on their ordering prescriptions. Moreover and most remarkably, the difference between these two correlators is reflected by the difference between $--$ and $+-$ correlations. It thus follows from [96] that the "surprising" effects found in [71], i.e. the presence of particle–antiparticle Bose–Einstein type correlations and a new term in the Bose–Einstein correlation function for neutral particles are reobtained, but under a more general form which contains also the quantum corrections. These equations also prove that the above effects are not an artefact of the classical current formalism but have general validity.

Moreover and most remarkably, from the above equations follows that *the difference between the effects of the classical and quantum currents resides in just these "new effects" and in particular in the difference between* 00 *and* $--$ *correlations, i.e. in the* $+-$ *correlations*. This result can serve as an estimate of the importance of quantum corrections to the classical current formalism of BEC. Since $+-$ correlations are in general small, it follows that the classical current approach is a good approximation, except for very short-lived sources, where the $+-$ correlations become comparable to the $--$ correlations. It also follows that the experimental measurement of $+-$ correlations is a highly rewarding task, since they are a rather unique tool for the investigation of two very interesting effects in BEC, namely squeezed states and quantum corrections.

4.8. Space–time form of sources in the classical current formalism

In [3] two types of sources were considered, a "static" one which corresponds to a source in rest and an expanding one. We will present below some of the results, as they exemplify certain important features of the space–time approach within the classical current formalism.

A static source: The space–time distributions of static sources, as well as the primordial correlator, are parametrised as Gaussians:

$$f_{\mathrm{ch}}(x) = \exp(-x_0^2/R_{\mathrm{ch},0}^2 - x_\parallel^2/R_{\mathrm{ch},\parallel}^2 - x_\perp^2/R_{\mathrm{ch},\perp}^2)\ , \tag{4.83}$$

$$f_{\mathrm{c}}(x) = \exp(-x_0^2/R_{\mathrm{c},0}^2 - x_\parallel^2/R_{\mathrm{c},\parallel}^2 - x_\perp^2/R_{\mathrm{c},\perp}^2)\ , \tag{4.84}$$

$$C(x-y) = \exp[-(x_0-y_0)^2/2L_0^2 - (x_\parallel - y_\parallel)^2/2L_\parallel^2 - (\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)^2/2L_\perp^2]\ . \tag{4.85}$$

Note that the term *static* here does not imply time independence but rather a specific time dependence defined by Eqs. (4.83) and (4.84) corresponding to source elements being at rest. This is to be contrasted to the *expanding* source, discussed in the next section, which explicitly contains velocities of source elements.

The main justification for this particular form of parametrisation is mathematical convenience, because, as will be shown below, for this case the correlation functions in momentum space can be calculated analytically and the physical implications can be read immediately.

In Eqs. (4.83)–(4.85), $R_{\mathrm{ch},\alpha}$ and $R_{\mathrm{c},\alpha}(\alpha = 0, \perp, \parallel)$ are the lifetimes, transverse radii and longitudinal radii of the chaotic source and of the coherent source, respectively, and $L_\alpha(\alpha = 0, \perp, \parallel)$ are the correlation time and the corresponding correlation lengths in transverse and in longitudinal direction. The relative contributions of the chaotic and the coherent component are determined by fixing the value of the (momentum dependent) chaoticity parameter p at some arbitrary scale (in this case, at $k=0$):

$$p_0 \equiv p(k=0)\ . \tag{4.86}$$

The model contains 10 independent parameters: the radii and lifetimes of the chaotic and of the coherent source, the correlation lengths in space and time, and the chaoticity p_0. In [3] it is assumed that $L_\parallel = L_\perp \equiv L$, i.e., that the medium is isotropic, which leaves us with nine independent parameters.

With the definitions

$$R^2_{\alpha L} = R^2_{\mathrm{ch},\alpha} L^2_\alpha / (R^2_{\mathrm{ch},\alpha} + L^2_\alpha) \quad (\alpha = 0, \perp, \parallel) \tag{4.87}$$

one may write the single inclusive distribution in the form

$$E(1/\sigma)\mathrm{d}^3\sigma/\mathrm{d}^3k = E(1/\sigma)\mathrm{d}^3\sigma/\mathrm{d}^3k|_{k=0}(p_0 s_{\mathrm{ch}}(k) + (1-p_0)s_{\mathrm{c}}(k))\ , \tag{4.88}$$

where

$$s_{\mathrm{ch}}(k) = \exp[-E^2R^2_{0L}/2 - k^2_\parallel R^2_{\parallel L}/2 - \boldsymbol{k}^2_\perp R^2_{\perp L}/2]\ , \tag{4.89}$$

$$s_{\mathrm{c}}(k) = \exp[-E^2R^2_{\mathrm{c},0}/2 - k^2_\parallel R^2_{\mathrm{c},\parallel}/2 - \boldsymbol{k}^2_\perp R^2_{\mathrm{c},\perp}/2]\ . \tag{4.90}$$

The scales which determine the mean energy–momentum of the coherently produced particles are given by the inverse lifetime and radii, $R^{-1}_{\mathrm{c},\alpha}$, of the coherent source. For the chaotically produced particles, these scales are given by the inverse of a combination of correlation lengths and dimensions of the chaotic source, $R^{-1}_{\alpha L}$. Eq. (4.87) implies that $R_{\alpha L} \le R_{\mathrm{ch},\alpha}$. The radius of the chaotic source enters the single inclusive distribution only in combination with the correlation length L. This feature which occurs also for higher-order correlations leads to the important consequence that experimental measurements of BEC do not provide separately information about radii (lifetimes) of sources, nor about correlation lengths (-times), but rather about the combination of these quantities as given by Eq. (4.87). On the other hand, by measuring both the single and the double inclusive distribution one can disentangle radii from correlation lengths.

It follows from Eqs. (4.88)–(4.90) that in the presence of partial coherence in general (i.e., unless $R_{\alpha L} = R_{\mathrm{c},\alpha}$) the single inclusive distribution is a superposition of two Gaussians of different widths. If the geometry of the coherent source is the same as that of the chaotic source, one has $R_{\mathrm{c},\alpha} = R_{\mathrm{ch},\alpha} > R_{L\alpha}$, which would imply that coherently produced particles can be observed predominantly in the soft regime. However, if the coherent radii are small compared to the chaotic ones, this situation is reversed.

As a next step, consider the correlation functions. The correlation function of two negatively charged pions is

$$C_2^{--}(k_1,k_2) = 1 + 2\sqrt{p_1(1-p_1)\cdot p_2(1-p_2)}\,T_{12}\cos(\phi^{\mathrm{ch}}_{12} - \phi^{\mathrm{c}}_1 + \phi^{\mathrm{c}}_2) + p_1p_2T^2_{12}\ . \tag{4.91}$$

For the Gaussian parametrisations all phases in the second-order correlation function disappear,[34]

$$\phi^{\mathrm{ch}}_{12} = \tilde{\phi}^{\mathrm{ch}}_{12} = \phi^{\mathrm{c}}_j = \tilde{\phi}^{\mathrm{c}}_j = 0\ , \tag{4.92}$$

[34] This is not the case anymore for an expanding source, e.g. (see below) or in general for higher-order correlations.

and

$$T_{12} = \exp\left[-\frac{(E_1 - E_2)^2(R^2_{\mathrm{ch},0} - R^2_{0L})}{8} - \frac{(k_{1,\parallel} - k_{2,\parallel})^2(R^2_{\mathrm{ch},\parallel} - R^2_{\parallel L})}{8} - \frac{(\boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp})^2(R^2_{\mathrm{ch},\perp} - R^2_{\perp L})}{8} \right], \tag{4.93}$$

$$\tilde{T}_{12} = \exp\left[-\frac{(E_1 + E_2)^2(R^2_{\mathrm{ch},0} - R^2_{0L})}{8} - \frac{(k_{1,\parallel} + k_{2,\parallel})^2(R^2_{\mathrm{ch},\parallel} - R^2_{\parallel L})}{8} - \frac{(\boldsymbol{k}_{1\perp} + \boldsymbol{k}_{2\perp})^2(R^2_{\mathrm{ch},\perp} - R^2_{\perp L})}{8} \right]. \tag{4.94}$$

The two-particle correlation function C_2^{--} is the sum of a purely chaotic term ($\propto T^2_{12}$) and a mixed term ($\propto T_{12}$). The momentum dependence of the chaoticity parameter, $p = p(k)$, implies a momentum dependence of the contribution of the mixed term relative to that of the purely chaotic term. To see how this affects the interplay between the two terms (i.e., the interplay between the two Gaussians), it is useful to explicitly insert the momentum dependence of the chaoticity parameter by writing

$$p_r = p(k_r) = p_0/A_r \quad (r = 1,2) \tag{4.95}$$

with

$$A_r \equiv A(k_r) = p_0 + (1 - p_0)S_{rr} \quad (r = 1,2)\ , \tag{4.96}$$

and

$$S_{rs} = \exp\left[-\frac{(E_r^2 + E_s^2)(R^2_{c,0} - R^2_{0L})}{4} - \frac{(k^2_{r\parallel} + k^2_{s\parallel})(R^2_{c,\parallel} - R^2_{\parallel L})}{4} - \frac{(\boldsymbol{k}^2_{r\perp} + \boldsymbol{k}^2_{s\perp})(R^2_{c,\perp} - R^2_{\perp L})}{4} \right]. \tag{4.97}$$

With this, C_2^{--} takes the form

$$C_2^{--}(k_1, k_2) = 1 + [(2p_0(1 - p_0)S_{12}/A_1A_2)]T_{12} + (p_0^2/A_1A_2)T^2_{12}\ . \tag{4.98}$$

The momentum dependence of the relative contributions of the purely chaotic and of the mixed term is reflected in the factor S_{12}. Depending on the sign of the combinations $R^2_{c,\alpha} - R^2_{\alpha L}$, S_{12} may act either as a suppression factor or as an enhancement factor of the mixed term relative to the chaotic term. This is a consequence of the fact that, in contrast to the case of the correlation function C_2 derived within the wave-function formalism, where C_2 depends only on the difference of momenta $k_1 - k_2$ now the correlation function depends also on $k_1 + k_2$.[35]

[35] Till recently this desirable physical property, which is observed in most experimental data on BEC, was considered to be a consequence of the *expansion* of the source and used to be derived within the Wigner function formalism, which is also a particular case of the classical current formalism. As shown in the example treated above (see [33]), it can be considered also a consequence of the (partial) coherence of a non-expanding source.

It is instructive to discuss the tilde terms that give rise to the particle–antiparticle correlations for parametrisation (4.83)–(4.85) of a static source. For the sake of transparency, consider only the purely chaotic case, $p_0 = 0$. The correlation functions of like and unlike charged pions then take the form

$$C_2^{--}(k_1, k_2) = 1 + T_{12}^2 \;, \tag{4.99}$$

$$C_2^{+-}(k_1, k_2) = 1 + \tilde{T}_{12}^2 \;. \tag{4.100}$$

From (4.93) and (4.94) it can be seen that the "new" $\tilde{T}_{12}$ terms that appear in the particle–antiparticle correlations are in general small compared to the "ordinary" T_{12} terms that determine the particle–particle correlations. The term $\tilde{T}_{12}$ gives rise to an anticorrelation effect due to the factor in Eq. (4.94) containing the sum $\boldsymbol{k}_1 + \boldsymbol{k}_2$, if the first factor, containing $E_1 + E_2$, is not too small. The latter is possible, if the time duration of the pion emission process and/or the pion energies are sufficiently small. We thus expect an enhanced contribution of the "new" terms for soft pions. The appearance of anticorrelations[36] is, as mentioned above, a general property of squeezed states, which are present in the space–time formalism of [3]. The tilde terms arise as a consequence of the non-stationarity of the source. In the limit of a stationary source, $R_0 \to \infty$, and $\tilde{T}_{12} \to 0$. An upper limit is given by

$$C_2^{+-} = 1 + |\tilde{T}_{12}|^2 \le 1 + \exp[-(R_0^2 - R_{0L}^2)m_\pi^2] \;. \tag{4.101}$$

In the limit $L_0 \gg R_0$ on the other hand, $R_0 \simeq R_{0L}$ and C_2^{+-} reaches its maximum value $(C_2^{+-})_{\max} = 2$. We observe in the above equation that the contribution of the "tilde" terms increases with decreasing mass of the particles and reaches its maximum of 2 for massless particles at fixed and non-vanishing $R_0 - R_{0L}$. This is related to the observation made in the case of photon BEC where we saw that for unpolarised photons the maximum of C_2 is also 2. Indeed, the role which the charge degrees of freedom (+ , −) play in the case of pions is for unpolarised photons taken over by the spin. From this mass dependence or more general from the energy dependence of the anticorrelation follows that if the factor $E_1 + E_2$ in Eq. (4.94) could be decreased, an enhancement of the "new" terms would emerge. A possible mechanism for this could be the sudden transition mechanism considered in [16]. Indeed as shown in this reference (see also Eqs. (2.15) and (2.16)) for a chaotic source the correlator $\langle a(\boldsymbol{k}_1)a(\boldsymbol{k}_2)\rangle$ characteristic of the "new" terms turns out to be an increasing function of the parameter $r = \frac{1}{2}\log(E_a/E_b)$ where E_a, E_b are the energies of the particle in the vacuum and medium, respectively. Thus by allowing for medium effects, which in a certain sense is equivalent to an effective change of mass,[37] one can possibly enhance the anticorrelation effect[38] in BEC.

Expanding source. High-energy multiparticle dynamics suggests that the sources of produced particles are expanding. This property is reflected in particular in hydrodynamical models and also in string models. In terms of the current formalism this means that the correlators are velocity

[36] Some authors have recently called them "back-to-back" correlations.

[37] This particular possibility was suggested in [97]. Andreev [98] suggested a time evolution scenario for the medium effect.

[38] Anticorrelations in disoriented chiral condensates are considered in [99].

dependent. While many of the studies of Bose–Einstein correlations for expanding sources have followed, with slight variations, a Wigner function type of approach, the use of the Wigner approach is in general too restrictive and is recommendable only in the case where a full-fledged hydrodynamical description of the system is performed. In the present section, following [3], we shall therefore start with a more general discussion of the expanding source which is based on the space–time current correlator and the space–time form of the coherent component and which is not affected by the semi-classical and small q approximations inherent in the Wigner function approach.

We introduce the variables τ, η and $x_{\|}$, with

$$\tau = \sqrt{x_0^2 - x_{\|}^2}\,, \qquad \eta = \tfrac{1}{2}\ln\left[(x_0 + x_{\|})/(x_0 - x_{\|})\right] . \tag{4.102}$$

Here τ is the proper time, $x_{\|}$ the coordinate in the longitudinal direction (e.g. the collision axis in p–p reactions or the jet axis in e^+–e^- reactions) and η the space–time rapidity. An ansatz which is invariant under boosts of the coordinate frame in longitudinal direction will be considered (Fig. 8). Physically, this ansatz is motivated by the prejudice that the single inclusive distribution in rapidity is flat. The space–time distributions of the chaotic and of the coherent source and the correlator are then parametrised as

$$f_{\mathrm{ch}}(x) \sim \exp[-(\tau - \tau_{0,\mathrm{ch}})^2/(\delta\tau_{\mathrm{ch}})^2]\exp(-x_\perp^2/R_{\mathrm{ch}}^2) . \tag{4.103}$$

$$f_{\mathrm{c}}(x) \sim \exp[-(\tau - \tau_{0,\mathrm{c}})^2/(\delta\tau_{\mathrm{c}})^2]\exp(-x_\perp^2/R_{\mathrm{c}}^2) \tag{4.104}$$

$$\begin{aligned} &C(\tau_1 - \tau_2, \eta_1 - \eta_2, x_{\perp,1} - x_{\perp,2}) \\ &\quad = \exp[-(\tau_1 - \tau_2)^2/2L_\tau^2 - (2\tau_1\tau_2/L_\eta^2)\sinh^2((\eta_1 - \eta_2)/2) - (\boldsymbol{x}_{\perp,1} - \boldsymbol{x}_{\perp,2})^2/2L_\perp^2] . \end{aligned} \tag{4.105}$$

The model contains again 10 independent parameters: the proper time coordinates of the chaotic and the coherent source, $\tau_{0,\mathrm{ch}}$, $\tau_{0,\mathrm{c}}$, their widths in proper time, $\delta\tau_{\mathrm{ch}}$ and $\delta\tau_{\mathrm{c}}$, the transverse radii, R_{ch} and R_{c}, the correlation lengths $L_\tau, L_\perp$ and L_η, and the chaoticity parameter p_0.

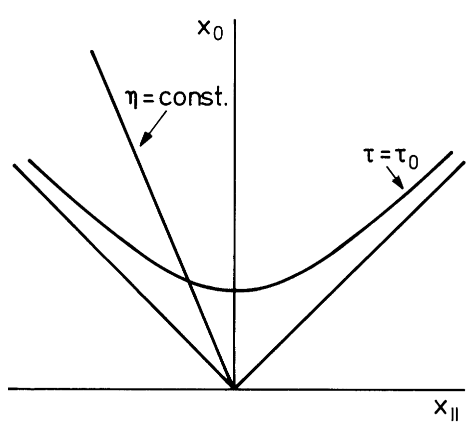


Fig. 8. Geometry of the boost-invariant source (from Ref. [3]).

In order to be able to obtain explicit expressions for the single inclusive distribution and the correlation functions, a further simplifying assumption is made, namely, that $\delta\tau_{ch} = \delta\tau_c = 0$. Eqs. (4.103) and (4.104) then take the form

$$f_{ch}(x) \sim \delta(\tau - \tau_{0,ch}) \exp(-x_\perp^2/R_{ch}^2) , \tag{4.106}$$

$$f_c(x) \sim \delta(\tau - \tau_{0,c}) \exp(-x_\perp^2/R_c^2) . \tag{4.107}$$

Now the results no longer depend on the correlation length L_τ, and one is left with seven independent parameters: $\tau_{0,ch}, \tau_{0,c}, R_{ch}, R_c, L_\perp, L_\eta$ and p_0.

The Fourier integrations necessary to obtain $D(k_1, k_2)$ and $I(k)$ can be performed by doing a saddle point expansion; this should provide a good approximation if

$$a_i \equiv m_{i\perp}\tau_{0,ch}/2 \gg 1, \qquad b \equiv \tau_{0,ch}^2/2L_\eta^2 \gg 1 . \tag{4.108}$$

With the definitions:

$$R_L^2 \equiv R_{ch}^2 L_\perp^2/(R_{ch}^2 + L_\perp^2) , \tag{4.109}$$

$$\gamma_{12} \equiv \tau_{0,ch}(m_{1\perp} - m_{2\perp})/L_\eta^2 m_{1\perp}m_{2\perp}, \qquad \tilde{\gamma}_{12} \equiv \tau_{0,ch}(m_{1\perp} + m_{2\perp})/L_\eta^2 m_{1\perp}m_{2\perp} , \tag{4.110}$$

the single inclusive distribution can be written as the sum of a chaotic and a coherent term

$$E(1/\sigma)\mathrm{d}^3\sigma/\mathrm{d}^3k = (p_0 s_{ch}(k) + (1-p_0)s_c(k))E(1/\sigma)\mathrm{d}^3\sigma/\mathrm{d}^3k|_{k=0} \tag{4.111}$$

with

$$s_{ch}(k) = (m_\pi/m_\perp)\exp[-\boldsymbol{k}_\perp^2 R_L^2/2] , \tag{4.112}$$

$$s_c(k) = (m_\pi/m_\perp)\exp[-\boldsymbol{k}_\perp^2 R_c^2/2] , \tag{4.113}$$

where $m_\perp$ is the transverse mass of the pions emitted. The momentum dependence of the chaoticity parameter takes the form

$$p_r = p(k_r) = p_0/A_r \quad (r = 1, 2) \tag{4.114}$$

with

$$A_r \equiv A(k_r) = p_0 + (1-p_0)S_{rr} \quad (r = 1, 2) , \tag{4.115}$$

and

$$S_{rs} = \exp[-(\boldsymbol{k}_{r\perp}^2 + \boldsymbol{k}_{s\perp}^2)(R_c^2 - R_L^2)/4] . \tag{4.116}$$

Unless $R_c = R_L$, the transverse momentum distribution is a superposition of two Gaussians of different widths. The rapidity distribution is uniform, $\mathrm{d}N/\mathrm{d}y = const.$, as a result of boost invariance. In opposition to what is assumed usually in simplified quasi-hydrodynamical treatments, the transverse radius of the chaotic source, R_{ch}, cannot be determined independently by measuring only the single inclusive distribution, as the quantity R_L which sets the scale for the mean transverse momentum of the chaotically produced particles is a combination of R_{ch} and the correlation length $L_\perp$.

We recall that the second-order correlation functions

$$C_2^{++}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + 2\sqrt{p_1(1-p_1)\cdot p_2(1-p_2)}T_{12}\cos(\phi_{12}^{\rm ch} - \phi_1^{\rm c} + \phi_2^{\rm c}) + p_1 p_2 T_{12}^2 \quad (4.117)$$

are defined in terms of the magnitudes and phases of d_{rs} and $\tilde{d}_{re}$, T_{rs}, $\tilde{T}_{rs}$, $\phi_{rs}^{\rm ch}$ and $\tilde{\phi}_{rs}^{\rm ch}$ of the chaotic source as well as of the phases of the coherent component, $\phi_r^{\rm c}$. The expressions for these quantities read for an expanding source:

$$T_{12} = (1+\gamma_{12}^2)^{-1/4}\exp[-[b/(1+\gamma_{12}^2)](y_1-y_2)^2 - (\boldsymbol{k}_{1\perp}-\boldsymbol{k}_{2\perp})^2(R_{\rm ch}^2 - R_L^2)/8] ,$$

$$\tilde{T}_{12} = (1+\tilde{\gamma}_{12}^2)^{-1/4}\exp[-[b/(1+\tilde{\gamma}_{12}^2)](y_1-y_2)^2 - (\boldsymbol{k}_{1\perp}-\boldsymbol{k}_{2\perp})^2(R_{\rm ch}^2 - R_L^2)/8] , \quad (4.118)$$

$$\phi_{12}^{\rm ch} = [b\gamma_{12}/(1+\gamma_{12}^2)](y_1-y_2)^2 - \tau_{0,\rm ch}(m_{1\perp}-m_{2\perp}) - \tfrac{1}{2}\arctan\gamma_{12} , \quad (4.119)$$

$$\tilde{\phi}_{12}^{\rm ch} = [b\tilde{\gamma}_{12}/(1+\tilde{\gamma}_{12}^2)](y_1-y_2)^2 - \tau_{0,\rm ch}(m_{1\perp}+m_{2\perp}) - \tfrac{1}{2}\arctan\tilde{\gamma}_{12} , \quad (4.120)$$

$$\phi_j^{\rm c} = -\tau_{0,\rm c}m_{j\perp} . \quad (4.121)$$

One thus finds again that the correlation functions do not depend separately on the geometrical radii R or on the correlation lengths L but rather on the combination R_L defined in (4.109). This expression reduces in the limit $R_{\rm ch} \gg L$ to L and in the limit. $R_{\rm ch} \ll L$ to R. The model considered in [39] is thus a particular case of the space–time approach [3] for $L = 0$.

As in the static case the tilde terms give rise to the particle–antiparticle correlations. For a purely chaotic system the intercept of the $\pi^+\pi^-$ correlation function is

$$C_2^{+-}(k,k) = 1 + (1+4(b/a)^2)^{-1/2} = 1 + (1+4(\tau_{0,\rm ch}/m_\perp L_\eta^2)^2)^{-1/2} . \quad (4.122)$$

We conclude this section with the observation that in [3] a correspondence between the correlation length L in the primordial correlator $C(x-y)$ and the temperature T for a pion source that exhibits thermal equilibrium was established. In the limit of large volume $V \propto R^3$ and lifetime R_0 of the system, it reads

$$L \sim T^{-1} . \quad (4.123)$$

4.9. The Wigner function approach

As mentioned previously, the experimental observation of the fact that the two particle correlation function depends not only on the difference of momenta $q = k_1 - k_2$ but also on the sum $k_1 + k_2$ led to the introduction and the use [100] of a "source" function within the well-known Wigner function formalism of quantum mechanics.[39] While it turned out later that this property of the correlation function can be derived within the current formalism without the semiclassical approximations involved by the Wigner formalism, this formalism is still useful when applied within a hydrodynamical context.

[39] An attempt to consider the correlation between coordinates and momentum was also performed earlier within the ordinary wave-function formalism by Yano and Koonin [94] who proposed a formula for the second-order correlation function of form (4.67). However, this form turned out subsequently to have pathological features as it leads in some cases to a violation of the lower bounds of the correlation function (see Section 5.1.6). The reason for this misbehaviour is mentioned in Section 4.6 and will also be discussed in the following.

The Wigner function approach for BEC was proposed in a non-relativistic form in Ref. [100] and subsequently generalised in [101,3] (see also [64,95]). The Wigner function called also source function, $g(x,k)$, may be regarded as the quantum analogue of the density of particles of momentum k at space–time point x in classical statistical physics. It is defined within the wave-function formalism as

$$\begin{aligned} g(\boldsymbol{x},\boldsymbol{k},t) &= \int d^3x'\psi^*\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{x}',t\right)\psi\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{x}',t\right)e^{i\boldsymbol{k}\boldsymbol{x}'} \\ &= \int d^3k'\psi^*\left(\boldsymbol{k}+\frac{1}{2}\boldsymbol{k}',t\right)\psi\left(\boldsymbol{k}-\frac{1}{2}\boldsymbol{k}',t\right)e^{-i\boldsymbol{k}'\boldsymbol{x}} \end{aligned} \tag{4.124}$$

and is related to the coordinate and momentum densities by the relations

$$n(\boldsymbol{x},t) = \int d^3k\, g(\boldsymbol{x},\boldsymbol{k},t)\ , \tag{4.125}$$

$$n(\boldsymbol{k},t) = \int d^3x\, g(\boldsymbol{x},\boldsymbol{k},t)\ , \tag{4.126}$$

respectively.

Due to its quantum nature the function $g(x,k)$ takes real but not necessarily positive values. Although Eq. (4.124) is nothing but a definition which does not imply any approximation, its form suggests that it might be useful when simultaneous information about coordinates and momenta are desirable, provided of course that the limits imposed by uncertainty relations are not violated. As a matter of fact as will be shown below, the Wigner function is useful for BEC only if a more stringent condition is fulfilled, namely that the difference of momenta q of the pair is small, as compared with the individual momenta of the produced particles. It is thus clear that its applicability is more restricted than that of the classical current approach, where only the "no recoil" condition, i.e. small total momentum of produced particles, as compared with the momentum of incident particles, must be respected. This circumstance is often overlooked when comparing theoretical predictions based on the Wigner approach with experimental data. In particular, it also follows that the application of the Wigner formalism to data has necessarily to take into account from the beginning resonances which dominate the small q region. It turns out that the use of the Wigner function for BEC is heuristically justified only in special cases as, e.g. when a coherent hydrodynamical study is performed, i.e. when the observables are related to an equation of state and when simultaneously single- and higher-order inclusive distributions are investigated. Unfortunately only very few papers, where the Wigner function formalism is used, are bonafide hydrodynamical studies. The majority of papers in this context are "quasi-hydrodynamical" (see Sections 5.1.5 and 5.1.6) in the sense that the form of the source function is expressed in terms of *effective* physical variables like temperature or velocity, which are not related by an equation of state. In this case the application of the Wigner approach is a "luxury" which is not justified. This is a fortiori true since, as will be shown in the following, the Wigner approach is mathematically not simpler than the classical current approach, of which it is a particular case. Thus the space–time model [3] presented above (see Section 4.8) is more general than the Wigner approach, albeit it is not more complicated and does not have more independent parameters.

In the second quantisation $g(x,k)$ is defined in terms of the correlator $\langle a^\dagger(\boldsymbol{k}_i)a(\boldsymbol{k}_j)\rangle$ by the relation

$$\langle a^\dagger(\boldsymbol{k}_i)a(\boldsymbol{k}_j)\rangle = \int d^4x \exp[-ix_\mu(k_i^\mu - k_j^\mu)]\cdot g[x, \tfrac{1}{2}(k_i + k_j)] \ . \tag{4.127}$$

This is a natural generalisation of (4.126) to which it reduces in the limit $k_i = k_j$.

Accordingly, for the second-order correlation function one writes

$$P_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \int d^4x_1 \int d^4x_2 [g(x_1,k_1)g(x_2,k_2) + g(x_1,K)g(x_2,K)\exp[iq_\mu(x_1^\mu - x_2^\mu)]] \ , \tag{4.128}$$

where $K^\mu = (k_1^\mu + k_2^\mu)/2$ and $q^\mu = k_1^\mu - k_2^\mu$ are the mean momentum and momentum difference of the pair.[40]

The relation between this Wigner approach and the classical current approach is established by expressing the r.h.s. of Eq. (4.127) in terms of the currents. One has

$$g(x,k) = \frac{1}{2\sqrt{E_iE_j}(2\pi)^3}\int d^4z \left\langle J\left(x+\frac{z}{2}\right)J\left(x-\frac{z}{2}\right)\right\rangle \exp[-ik^\mu z_\mu] \ . \tag{4.129}$$

The derivation of the Wigner formalism from the classical current formalism has the important advantage that it avoids violations of quantum mechanical bounds as those mentioned previously.

Note that in the r.h.s. of Eq. (4.128) enters the off-mass shell average momentum $\frac{1}{2}(k_1 + k_2)$ which is not equal to the on-mass shell average $K = \frac{1}{2}\sqrt{E^2 - m_1^2 + m_2^2}$ where E is the total energy of pair $(1,2)$. This means among other things that in this approach it is not enough to postulate the source function g in order to determine the second (and higher-order) correlation function C_2, but further assumptions are necessary. Usually one neglects the off-mass shellness, i.e. one approximates E by the sum $E_1 + E_2$ where E_i are the on-shell energies of particles $(1,2)$, which means that one neglects quantum corrections[41] which is permitted as long as $k_1 - k_2 = q$ is small.[42]

As mentioned already, the use of the Wigner formalism is worthwhile within a true hydrodynamical approach when the relation with the equation of state is exploited. In this case the probability to produce a particle of momentum k from the space–time point x depends on the fluid velocity, $u^\mu(x)$, and the temperature, $T(x)$, at this point, and one has

$$\sqrt{E_iE_j}\langle a^\dagger(\boldsymbol{k}_i)a(\boldsymbol{k}_j)\rangle = \frac{1}{(2\pi)^3}\int_\Sigma \frac{(1/2)(k_i^\mu + k_j^\mu)\, d\sigma_\mu(x_\mu)}{\exp[(1/2)(k_i^\mu + k_j^\mu)u_\mu(x_\mu)/T_f(x_\mu)] - 1}\cdot \exp[-ix_\mu(k_i^\mu - k_j^\mu)] \tag{4.130}$$

[40] For neutral particles, there are additional contributions to $P_2(\boldsymbol{k}_1,\boldsymbol{k}_2)$ which play a role for soft particles and which will be neglected here.

[41] That these corrections can be important has also been shown in [102].

[42] It is sometimes argued that the relevant q range in BEC is given by D^{-1} where D is a typical length scale of the source and therefore for heavy ion reactions this should be allowed. This is not quite correct, because the *shape* of the correlation function from which one determines the physical parameters of the source is not given just by the values of the correlation function near the origin, but depends also on its values at large q.

Here, $d\sigma^{\mu}$ is the volume element on the freeze-out hypersurface Σ where the final state particles are produced. We will discuss applications of this approach in Section 5.

4.9.1. Resonances in the Wigner formalism

For a purely chaotic source, the formalism in order to take into account the effects of resonance decays on the Bose–Einstein correlation function can be found, e.g. in [18,54]. An extension of this approach is due to [56,101] which allows to consider also the effect of coherence and provides rather detailed and subsequently, apparently, confirmed predictions for heavy ion reactions. It is based on the Wigner function formalism.

The correlation function of two identical particles of momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ can be written as

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + (A_{12}A_{21}/A_{11}A_{22})\ , \tag{4.131}$$

where the matrix elements A_{ij} are given in terms of source functions $g(x,k)$ as follows:

$$A_{ij} = \sqrt{E_i E_j}\langle a^{\dagger}(\boldsymbol{k}_i)a(\boldsymbol{k}_j)\rangle = \int d^4x g(x_\mu, k^\mu) e^{iq^\mu x_\mu}\ . \tag{4.132}$$

A typical source function reads

$$g(x_\mu, p^\mu) = g_\pi^{dir}(x_\mu, p^\mu) + \sum_{res=\rho,\omega,\eta,\dots} g_{res\to\pi}(x_\mu, p^\mu) \tag{4.133}$$

where the labels *dir* and $res \to \pi$ refer to direct pions and to pions which are produced through the decay of resonances (such as ρ, ω, η, etc.), respectively.

The contribution from a particular resonance decay is estimated in [56,101] using kinematical and phase space considerations as well as the source function of that resonance. The source distribution for the direct production of pions and resonances is calculated assuming local thermodynamical and chemical equilibrium as is appropriate for a hydrodynamical treatment.

$$g_\alpha^{dir}(x_\mu, p^\mu) = \frac{2J+1}{(2\pi)^3}\int_\Sigma \frac{p^\mu\, d\sigma_\mu(x'_\mu)\delta^4(x_\mu - x'_\mu)}{\exp[[p^\mu u_\mu(x'_\mu) - B_\alpha \mu_B(x'_\mu) - S_\alpha \mu_S(x'_\mu)]/T_f(x'_\mu)] - 1}\ . \tag{4.134}$$

Here α denotes the particular resonance and $d\sigma^\mu$ is the differential volume element and the integration is performed over the freeze-out hypersurface Σ. $u^\mu(x)$ and T_f are the four-velocity of the fluid element at point x and the freeze-out temperature, respectively. B and S are the baryon number and the strangeness of the particle species labelled α, respectively, and μ_B and μ_S are the corresponding chemical potentials. J is the spin of the particle.

This approach is then extended [56] to include also a coherent component resulting in a second-order correlation function of the form

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + 2p_{eff}(1 - p_{eff})\mathrm{Re}\, d_{12} + p_{eff}^2|d_{12}|^2\ , \tag{4.135}$$

where p_{eff} is an effective chaoticity related to the true chaoticity p_{dir} [43] via

$$p_{eff} = p_{dir}(1 - f^{res}) + f^{res}\ . \tag{4.136}$$

[43] It is assumed that only directly produced particles have a coherent component.

f^{res} is the fraction of particles arising from resonances. The form of this equation is the same as that derived previously for a partially coherent source within the current formalism and manifests the characteristic two-component structure.

The sensitivity of the correlation function on the chaoticity parameter p_{dir} can be estimated, e.g. from the intercept (see Eq. (4.135))

$$I_o = C_2(\boldsymbol{k},\boldsymbol{k}) = 1 + 2p_{\mathrm{eff}} - p_{\mathrm{eff}}^2 \ . \tag{4.137}$$

The fractions of pions produced directly (chaotically and coherently) and from resonances are

$$f_{\mathrm{ch}}^{dir} = p_{dir} A_{ii}^{\pi}/A_{ii}, \quad f_{\mathrm{co}}^{dir} = (1 - p_{dir}) A_{ii}^{\pi}/A_{ii}, \quad f^{res} = \Sigma_{res=\rho,\omega,\eta,\dots} A_{ii}^{res}/A_{ii} \tag{4.138}$$

with

$$f_{\mathrm{ch}}^{dir} + f_{\mathrm{co}}^{dir} + f^{res} = 1 \ .$$

In Fig. 9 the intercept of the correlation function is shown as a function of p_{dir} and f^{res}. In order to read off the fraction of *direct* chaotically produced particles, p_{dir}, from the intercept of the correlation function, one has to extract the effective chaoticity p_{eff} according to Eq. (4.137) and then correct for the fraction of pions from resonance decays. Note that $p_{\mathrm{eff}} < p_{dir}$. In particular, if a large fraction of pions arises from resonance decays, $p_{\mathrm{eff}} \to 1$ and one needs very

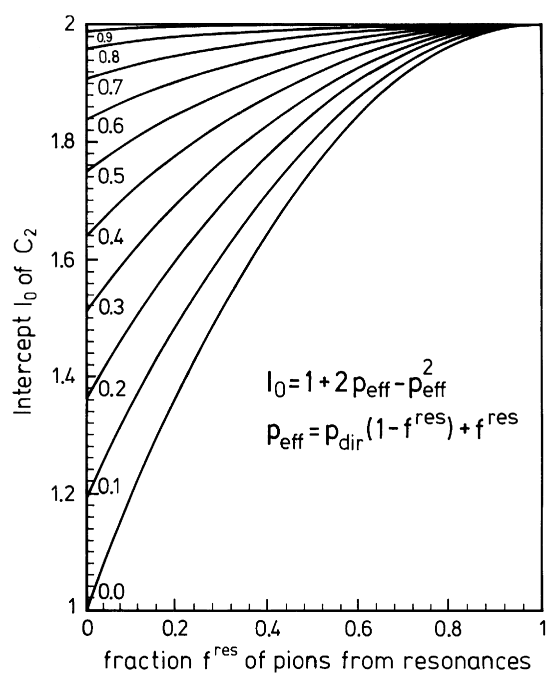


Fig. 9. Intercept of two-particle correlation function in the presence of coherence and resonances (from Ref. [56]).

precise measurements of the two-particle correlation function at small $\boldsymbol{q}$ to determine the true chaoticity, p_{dir}.

A further complication arises if a fraction of particles is the decay products of long-lived resonances.[44] This topic as well as the problem of misidentification are discussed in [56].

4.10. Dynamical models of multiparticle production and event generators

Due to the lack of a full-fledged theory of multiparticle production in strong interactions different models of multiparticle dynamics were proposed. Bose–Einstein correlations measurements have been used either to test a particular model or/and to determine some of its parameters. Among other things these models were used to predict the dependence of the chaoticity on the type of reaction. In the following, we will sketch the main theoretical ideas on which these models are based and mention briefly their relation to data.

One of the first models of particle production from which definite predictions on BEC can be derived is the Schwinger model [104] for e^+–e^- reactions. It visualises the source as an one-dimensional string in a coherent state and thus predicts the absence of any bunching effect. A similar prediction follows from the bremsstrahlung model [105]. Recoilless bremsstrahlung can be described by a classical current which also corresponds to a coherent state. Given the fact that in all hadron production processes BEC, i.e. a bunching effect has been seen, it follows that the above two models are ruled out by experiment.

More complex predictions follow from a *dual topological* model due to Giovannini and Veneziano [106] which associates the processes e^+–$e^- \to$ *hadrons* to a unitarity cut in one plane, reactions induced by Pomeron exchange to a cut in two planes, and annihilation reactions $\bar{p}$–p to a cut in three planes. This model predicts then among other things that for $\pi^-\pi^-$ BEC the intercepts $C_2(k,k)$ of the second-order correlation functions for the above reactions should satisfy the following relation:

$$[C_2^{e^+e^-}(k,k) - 2]/[C_2^{\pi p}(k,k) - 2]/[C_2^{ann}(k,k) - 2] = 1/\tfrac{1}{2}/\tfrac{1}{3}\ . \tag{4.139}$$

(A similar, but quantitatively different relationship is predicted for $\pi^+\pi^-$ correlations.)

Despite the fact that since the publication of this paper in 1977 many experimental BEC studies of these reactions have been performed, the above predictions could not be tested quantitatively in a convincing manner. This is due among other things to experimental difficulties (see Section 4.11) and illustrates the unsatisfactory status of experimental BEC investigations. A qualitative remark can however be made: the expectation that the annihilation reaction leads to more bunching than other reactions is apparently confirmed (see e.g. Ref. [12] and Section 2.1.1). As to the difference between e^+–e^- reactions and hadronic reactions the experimental situation is rather confused (see also below).

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[44] In some papers [103] pions originating from long-lived resonances are associated with a "halo" while those coming form short-lived resonances or directly produced are related to a "core". Then it is claimed among other things that the "core" parameters of the source (like radius and λ factor) can be obtained from the data just by eliminating the small Q points and fitting only the remaining points. Even if such a separation would be clear cut (there are doubts about this because of the ω resonance), it would be of course dependent on the resolution of the detector.

A somewhat related dynamical model based on Reggeon theory, has been already proposed in [107]. A straightforward extension of this formalism to heavy ion reactions does not work as it predicts that the longitudinal radius is of "hadronic" size [108].

A different approach to BEC based on the classical current formalism is proposed in [109]. The currents are associated with the chains of the dual parton model, and contrary to what is assumed in other applications of the classical current formalism, all the phases of these elementary currents are fixed, so that the source is essentially coherent. This is a special case of the classical current approach presented in Sections 4.2, 4.3 and 4.8 where allowance is made both for a chaotic and coherent component. The model is intended to work for p–p reactions where the authors state that resonances do not play an important role. It explains, according to the authors, the dependence of the λ parameter in the empirical formula for the second-order correlation function

$$C_2 = 1 + \lambda \exp(-R^2 q^2) \tag{4.140}$$

on the multiplicity and energy. Unfortunately, the claim that in p–p reactions pions are only directly produced is unfounded. Furthermore, there are other factors which influence the multiplicity dependence of λ (see Section 6.2) which are not considered in [109] and which are of more general nature.

An orthogonal point of view for the interpretation of the same λ factor (also for directly produced pions, only,) is due to [110]. In this approach the source is made of totally chaotic elementary emitting cells which are occupied by identical particles subject to Bose–Einstein statistics. Different cells are independent so that correlations between particles in different cells lead to $\lambda = 0$, while correlations between particles in the same cell are characterised by $\lambda = 1$. From the interplay of these two types of correlations, one obtains with an appropriate weighting, large λ values in e^+–e^- reactions and small λ values in p–p reactions, as in [109], but within a completely different approach.

We conclude the discussion of these two approaches by the following remarks. Besides the reservations about the role of directly produced pions in BEC expressed above and which presents the two approaches in a rather academic light, it is unclear whether the λ factor in e^+–e^- reactions is larger than in p–p reactions as assumed in [110]. This issue awaits a critical analysis of the specific experimental set-ups. The fact that quite different approaches lead to similar conclusions about the λ factor confirms that the parametrisation of the second-order correlation function in the form (4.140) is (see also Section 2.2) inadequate.

We discuss now other two, closely related, approaches, which make more detailed predictions about the form of the correlation function in e^+–e^- reactions: Refs. [22,75] on the one hand, and Refs. [23,111,121] on the other. Both approaches are based on a variant of the string model (for a more extended review of this topic see e.g. [113]). Such a string represents a coloured field formed between a quark q and an antiquark $\bar{q}$, which tend to separate. Because of confinement the break-up of the string can be materialised only through creation of new $q\bar{q}$ pairs, which are the mesons produced in the reaction.

The difference between the Schwinger model of confinement on the one hand and the models of Bowler and Andersson–Hofmann–Ringnér on the other is that in the former the field couples directly and locally to a meson, while in the latter ones the quarks, which constitute the meson, are created at different points. This feature destroys the coherence inherent in the Schwinger model and makes the Bose–Einstein bunching effect possible. For massless quarks the second-order correlation function can be approximated by the relation [23]

$$C_2 = 1 + \langle \cos(\kappa \Delta A)/\cosh(b\Delta A/2) \rangle , \tag{4.141}$$

where ΔA denotes the difference between the space–time areas of coloured fields spanned by the two particles, κ is the string tension and b a parameter characterising the decay probability of the string. For massive quarks the formulae become more involved and were approximated analytically in [22] or calculated numerically in [23,111,112]. In this model the correlation function depends both on the difference of momenta $k_1 - k_2$ as well as on their sum $k_1 + k_2$, reflecting the correlation between the momentum of the particle and the coordinate of its production point. This is a consequence of the fact that string models use a Wigner-function-type approach. From the above equations it follows that there are two length scales in the problem, one associated with κ and the other with b. Phenomenologically these correspond to $q_\parallel$ and $q_\perp$. Both these lengths are correlation lengths rather than geometrical radii. (As a matter of fact there is no geometrical radius in the string model.) Their magnitudes are quite different.

In both string approaches one obtains a difference between BEC for identically charged and neutral pions as found in [71]. However while in [22] there is room for coherence, this is apparently not the case for [23,111,112], which predict a totally chaotic source. Furthermore in [22] an energy dependence of the BEC is predicted (the correlation function is expected to shrink with increasing energy), while in [23,111,112] the correlation function does not depend on energy.[45]

A rather discordant note in this string concerto [22,23] is represented by the paper by Scholten and Wu [114]. These authors, using a different hadronisation mechanism conclude that dynamical correlations, at least in e^+–e^- reactions dominate over BEC correlations so that BEC cannot be used to infer information about the size and lifetime of the source.[46] This point of view seems too extreme, as it is contradicted by some simple empirical observations: in e^+–e^- reactions, as well as in all other reactions, correlations between identical particle are observed which are much stronger than those of non-identical ones, the correlation functions are (in general) monotonically decreasing functions of the momentum difference q and in nuclear reactions the "radii" obtained from identical particle correlations increase with the mass number of the participating nuclei. All these observations are in agreement with what one would expect from BEC, which suggests that dynamical correlations cannot distort this picture too much. However a thorough comparison of BEC in different reactions, using the same experimental techniques, appears highly desirable.

Event generators. The model [23] was implemented by Sjostrand [115] into JETSET under the name LUBOEI by modifying a posteriori the momenta of produced pions so that identical pairs of pions are bunched according to [23]. This manipulation "by hand" violates energy–momentum

[45] To make the model more realistic in Ref. [23] resonances were included according to the variant of the Lund model (JETSET) in use at that time (1986) and agreement with e^+–e^- data was found. Subsequently however it was pointed out in [58] that some resonance weights used in [23] were incorrect, so and the agreement mentioned above was probably accidental.

[46] What concerns e^+–e^- reactions similar scepticism was expressed by Haywood [4].

conservation which was imposed at the beginning in JETSET. To compensate for this, the momenta are rescaled so that energy–momentum conservation is restored. However this rescaling introduces spurious long-range correlations, which bias the BEC. Nevertheless, in general this program leads to a reasonable description of the bunching effect in the second-order correlation function.[47] More refined features of BEC which reflect the quantum mechanical essence of the effect, cannot be obtained of course. One reason for this, of rather technical nature, is due to the fact that the ad hoc modification of two-particle correlations does not yet include many-body correlations, reflected in the symmetrisation (or antisymmetrisation) of the entire wave function.

Another reason of fundamental character is that event generators like any Monte Carlo algorithm deal in general with probabilities[48] and therefore cannot account for quantum effects, which are based on phases of amplitudes.[49] The event generator JETSET was further developed by Lönnblad and Sjöstrand [118,119] and used to estimate the influence of BEC on the determination of the mass of W in e^+–e^- reactions, a subject of high current interest for the standard model and in particular for the search of the Higgs particle. This effect was also studied using different event generators in [120,121]. The argument of Lönnblad and Sjöstrand is the following. Consider the reaction e^+–$e^- \to W^+W^- \to q_1\bar{q}_2q_3\bar{q}_4$ when both W's decay into hadrons. Then according to [122] the typical space–time separation of the decay vertices of the W^+ and the W^- is less than 0.1 fm (at LEP 2 energies) and thus much smaller than a typical hadronic radius (~ 0.5 fm). There will thus be a Bose–Einstein interference between a pion from the W^+ and a pion (with the same charge) from the W^- and one cannot establish unambiguously the "parenthood" of these pions. This prevents then in this model a precise determination of the invariant mass of the W's. In [118] algorithms for the inclusion of this effect into the determination of the mass of the W are proposed and for certain scenarios mass corrections of the order of 100 MeV at 170 GeV c.m. energy are obtained. However, as emphasised in [118] other scenarios with less or no effect of BEC on the mass determination of the W are possible. Thus in [120,121] effects of the order of only 20 MeV are found. For more details we refer the reader to the original literature.

This aspect of BEC is interesting in itself as it illustrates the possible applications of this effect in electroweak interactions, a domain which is beyond the usual application domain of BEC, i.e. that of strong interactions.

The Lund model was applied also to heavy ion reactions and then extended to include BEC (e.g. the SPACER version [55] of the Lund model). The topic of event generators for heavy ion reactions is of current interest because of the ongoing search for quark–gluon plasma. Padula et al. [95] suggested to use for this purpose the Wigner function formalism in order to take into account explicitly the correlation between momenta and coordinates, as implied by the inside–outside cascade approach. This is evidently another way of expressing the non-stationarity of the correlation function mentioned above. (An explicit introduction of momentum–coordinate correlations in

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[47] See, however, Ref. [116].

[48] For heavy ion reactions the "quantum molecular dynamics" (QMD) model [117] attempts to surpass this deficiency by using wave functions rather than probabilities as input. However, this model also neglects (anti)symmetrisation effects and cannot be used for interferometry studies.

[49] It is interesting to mention that for *one* string Andersson and Hofmann [23] proposed a formulation of the BEC effect in terms of *amplitudes*. However this procedure cannot be used to generate events.

particle physics event generators like JETSET/LUBOEI is not necessary, because the non-stationarity is delivered "free house" by the string model used in the LUND generator.)

On the other hand, the Wigner formalism may present also another advantage as emphasised more recently by Bialas and Krzywicki [123]. This has to do with the important difficulty mentioned above and which is inherent in all event generators, namely the probabilistic nature of Monte Carlo methods. The Wigner function has in certain limits the meaning of a wave function and thus provides quantum amplitudes. The proposal of Bialas and Krzywicki consists then in starting from the single-particle distribution $\Omega_0(\boldsymbol{k})$ constructed from non-symmetrised particle wave functions as produced by conventional event generators and writing the Wigner function

$$g(\boldsymbol{k};\boldsymbol{x}) = \Omega_0(\boldsymbol{k}) w(\boldsymbol{k};\boldsymbol{x})\,, \tag{4.142}$$

where $w(\boldsymbol{k};\boldsymbol{x})$ is the conditional probability that given that the particles with momenta $k_1, k_2, \dots, k_n$ are present in the final state, they are produced at the points $x_1, \dots, x_n$. Then the art of the model builder consists in guessing the probability $w(\boldsymbol{k};\boldsymbol{x})$. This may be easier than guessing from the beginning the exact Wigner function. For example, a simple ansatz would be to assume that the likelihood to produce a particle from a given space point is statistically independent of what happens to other particles. This means that $w(\boldsymbol{k};\boldsymbol{x})$ can be factorised in terms of the individual particles. Implementations of this scheme were discussed in [124,125].

When using the Wigner formalism or any model (like those used in event generators) which specifies momenta and coordinates simultaneously, one must of course watch that the correlations between momenta and coordinates do not become too strong. This apparently has not always been done.[50]

That such a procedure is dangerous since it can lead to unphysical *antibunching* effects, i.e. to the violation of unitarity was already mentioned in [89] (see Section 4.6). This point has been reiterated recently, e.g. in [127–129].

Concluding this section one should emphasise that event generators are just an experimental tool, sometimes useful in the design of detectors or for getting rudimentary information about experimentally inaccessible phase. Often they are however abused, e.g. to search for "new" phenomena: if agreement between data and event generators is found, one states that no "new" physics was found. Such a procedure is unjustified, because agreement with a model or an event generator is often accidental. Furthermore, for the reasons mentioned above event generators cannot be used to obtain the "true" correlation function, i.e. they are no substitute for a bonafide HBT experiment (see also [130] for a critical analysis of transport models from the point of view of interferometry).

4.11. Experimental problems

The confrontation of model predictions with experimental BEC data has been hampered by two major facts: (i) most models are idealisations, i.e. they use assumptions which are too strong. Examples of such assumptions are: neglect of final state interactions, boost invariance, particular

[50] E.g. it is unclear to us whether the "pure" multiple scattering approach of [126] satisfies the above constraint.

analytical forms of the correlation functions; (ii) for various reasons in almost no BEC experiment so far a "true" or "complete" correlation function was measured, i.e. a correlation function as defined by

$$C_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = P_2(\boldsymbol{k}_1, \boldsymbol{k}_2)/P_1(\boldsymbol{k}_1)P_1(\boldsymbol{k}_2)\,. \tag{4.143}$$

Here P_2 and P_1 are the double and single inclusive cross sections, respectively. What is measured usually is instead a function which differs from Eq. (4.143) in several respects.

The normalisation of C_2 is not done in terms the product of single inclusive cross sections P_1 but in terms of a "background" double inclusive cross section, which is obtained either by considering pairs of (identically charged) particles which come from different events, or by considering oppositely charged particles, or by simulating P_2 with an event generator which does not contain BEC.

One does not (yet) measure the full correlation function C_2 in terms of its six independent variables, but rather projections of it in terms of single variables like the momentum difference q, rapidity difference $y_1 - y_2$, etc.

Last but not least, the intercept of the correlation function, which contains important information about the amount of coherence, cannot be really measured at present because (a) one does not yet control sufficiently well the final state interactions which contribute to the intercept; (b) its experimental determination implies an extrapolation to $q = 0$. Such an extrapolation can be performed only if the analytical form of the correlation function at $q \geq 0$ is known, which is not the case.

For these reasons at present it is difficult to test quantitatively a given model, except when its predictions are very clear cut. This circumstance limits certainly the usefulness of BEC as a tool in determining the exact dynamics of a reaction.

5. Applications to ultrarelativistic nucleus–nucleus collisions

5.1. BEC, hydrodynamics and the search for quark–gluon plasma

The use of BEC in the search for quark–gluon plasma is in most cases based on hydrodynamics. This is so because the space–time evolution of the system can be assumed to be given by the equations of hydrodynamics the solutions of which are different depending whether a QGP is formed or not. In this way, hydrodynamics also provides information about the equation of state (EOS). QGP being a (new) *phase* it is described by a specific EOS which is different from that of ordinary hadronic matter. The proof that this phase has been seen must include information about its EOS and thus the combination of hydrodynamics with BEC constitutes the only consistent way through which the formation of QGP can be tested.

QCD predicts that the phase transition from hadronic matter to QGP takes place only when a critical energy density is exceeded. To measure this density we need to know the initial volume of the system. While via photon interferometry (see Sections 4.5 and 4.6) one can in principle measure the dimensions (and thus the energy density) of the initial state, hadron interferometry yields information only about the final freeze-out stage when hadrons are created. To obtain information about the initial state with hadronic probes, hydrodynamical models have to be used in order to extrapolate backwards from the final freeze-out stage where hadrons are created, to the interesting initial stage. The lifetime of the system as given by BEC is also an important piece of information for QGP search. Indeed, in order to decide whether we have seen the new phase, we have to measure the lifetime of the system. Only lifetimes exceeding significantly typical hadronic lifetimes (10^{-23} s) could prove the establishment of QGP (see below).

5.1.1. General remarks about the hydrodynamical approach

Besides the main advantages of hydrodynamics related to the information about initial conditions, freeze-out and equation of state, for the study of BEC in particular hydrodynamics is very useful because it provides the single inclusive distributions which are intimately connected with higher-order distributions as well as the weights and the space–time and momentum distributions of resonances, which strongly influence the correlations. The phenomenological applications of the hydrodynamical approach to data are however hampered by two circumstances.

(i) While the ultimate goal of BEC is the extraction of the minimum set of parameters which include radii and coherence lengths both for the chaotic and coherent components of the source, in practice, mainly because of limited statistics (but also because of an inadequate analysis of the data) one has to limit oneself to the determination of a reduced number of parameters, which we call in the following "effective radii" R_{eff} and "effective chaoticity" p_{eff}. In reality, R_{eff} is a combination of correlation lengths[51] L and geometrical lengths R as introduced in Sections 4.3 and 4.8. Only in the particular case where one length scale is much smaller than the other, can one assume that one measures a "pure" radius or a "pure" correlation length. For simplicity, in the following, we shall assume that this is the case and in particular we assume that $R \gg L$ so that, R_{eff} reduces to L. This limit might perhaps correspond to what is seen in experiment, if one considers the expansion of the system in the hadronic phase. (In the high-temperature limit $L \approx T^{-1}$, see Section 4.8).

(ii) The presentation of the data is still biased by theoretical prejudices. Instead of a consistent hydrodynamical analysis, much simplified models are used (see Section 5.1.5 where these models are presented under the generic name of quasi-hydrodynamics) for this presentation and therefore to obtain the real physical quantities, one would have to solve a complicated mathematical "inverse" problem, i.e. one would have to reconstruct the raw data from those presented in the experimental papers and then apply the correct theoretical analysis to these. This has not been done so far and even if the statistics is sufficient for this purpose, the outcome is questionable because of the difficulties implied by the numerics. (That is why it would be desirable that experimentalists and theorists perform a joint analysis of the data or at least that the data should be presented also in "raw" form.)

The nearest approximation to the solution of the "inverse" problem found in the literature, is that of [101,150,131] based on the application of the HYLANDER code by the Marburg group: It consists in fitting the results of the hydrodynamical calculation to the Gaussian form used by experimentalists:

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \lambda \exp[-\tfrac{1}{2}q_{\parallel}^2 R_{\parallel}^2 - \tfrac{1}{2}q_{\mathrm{out}}^2 R_{\mathrm{out}}^2 - \tfrac{1}{2}q_{\mathrm{side}}^2 R_{\mathrm{side}}^2] \equiv 1 + \lambda \exp\left(-\frac{1}{2}\sum (qR)^2\right) \quad (5.1)$$

[51] In the following the concept of length refers to space–time.

and comparing with the inverse width of the correlation function as presented in the experimental papers.[52] Here $q^\mu \equiv p_1^\mu - p_2^\mu$, $K^\mu \equiv \frac{1}{2}(p_1^\mu + p_2^\mu)$ and $q_\parallel$ and $K_\parallel$ denote the components of $\boldsymbol{q}$ and $\boldsymbol{K}$ in beam direction, and $q_\perp$ and $K_\perp$ the components transverse to that direction; q_{out} is the projection of the transverse momentum difference, $\boldsymbol{q}_\perp$ on the transverse momentum of the pair, $2\boldsymbol{K}_\perp$, and q_{side} the component perpendicular to $\boldsymbol{K}_\perp$. (For a source with cylindrical symmetry, the two-particle correlation function can be expressed in terms of the five quantities $K_\parallel$, $K_\perp$, $q_\parallel$, q_{side} and q_{out}.) $R_\parallel$, R_{side}, R_{out} are effective parameters, associated via Eq. (5.1) to the corresponding q components.

Eq. (5.1) is equivalent to an expansion of the correlation function $C(\boldsymbol{q},\boldsymbol{K})$ for small q. The use of Eq. (5.1) for the representation of correlations data implies then that one does not measure the geometrical radius of the system but the length of homogeneity, which means that energy density determinations based on BEC are an overestimate.[53] To take into account the fact that the correlation function depends in general not only on the momentum difference $q = k_1 - k_2$ but also on the sum $K = \frac{1}{2}(k_1 + k_2)$, the parameters R and λ are assumed to be functions of K and rapidity $\frac{1}{2}(y_1 + y_2)$.

Hadron BEC refer to the freeze-out stage. This stage is usually described by the Cooper–Frye formula [136]:

$$E\frac{dN}{d\boldsymbol{k}} = \frac{g_i}{(2\pi)^3}\int_\sigma \frac{p_\mu d\sigma^\mu}{\exp((p_\mu u^\mu - \mu_s - \mu_b)/T_f) - 1}\,, \tag{5.3}$$

which describes the distribution of particles with degeneracy factor g_i and four-momentum p^μ emitted from a hypersurface element $d\sigma^\mu$ with four-velocity u^μ.[54] After the cascading of the resonances we obtain the final observable spectra.

5.1.2. Transverse and longitudinal expansion

The equations of hydrodynamics are non-linear and therefore good for surprises. An illustration of this situation is represented by the realisation, described in more detail below, that the naive

[52] In [132] it was recommended that experimentalists should use the more complete formula

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \lambda\exp[-\tfrac{1}{2}q_\parallel^2 R_\parallel^2 - \tfrac{1}{2}q_{\text{out}}^2 R_{\text{out}}^2 - \tfrac{1}{2}q_{\text{side}}^2 R_{\text{side}}^2 - 2q_{\text{out}}q_\parallel R_{\text{out}\parallel}^2]\,, \tag{5.2}$$

where $2q_{\text{out}}q_\parallel R_{\text{out}\parallel}^2$ is called "cross" term. For a more detailed discussion of its meaning and dependence on the coordinate system, see. Ref. [133]. Of course, in view of the deficiencies of the entire phenomenological procedure outlined above and discussed in greater length in Section 5.1.5, these details are of limited importance. In particular they do not affect the conclusions discussed here.

[53] In [134,135] a distinction is made between the "local" length of homogeneity $L_h(x,k)$ and the "hydrodynamical" length, $\bar{L}_h(x)$ which is the ensemble average of the former.

[54] In most applications particles produced with momenta p_μ pointing into the interior of the emitting isotherm ($p_\mu d\sigma^\mu < 0$) were assumed to be absorbed and therefore their contribution to the total particle number was neglected. In Ref. [137] this effect was indeed estimated to be negligible and recent attempts to reconsider it could not change this conclusion. Another effect is the interaction of the freeze-out system with the rest of the fluid. This effect can be estimated by comparing the evolution of the fluid with and without the frozen-out part. This is done by equating the frozen-out part with that corresponding in the equation of hydrodynamics to the case $p_\mu = 0$. The fluid parameters are modified by this procedure at a level not exceeding 10% [138]. The influence of the freeze-out mechanism on the determination of radii via BEC has been discussed recently in several papers; see. e.g. [139] and references quoted therein.

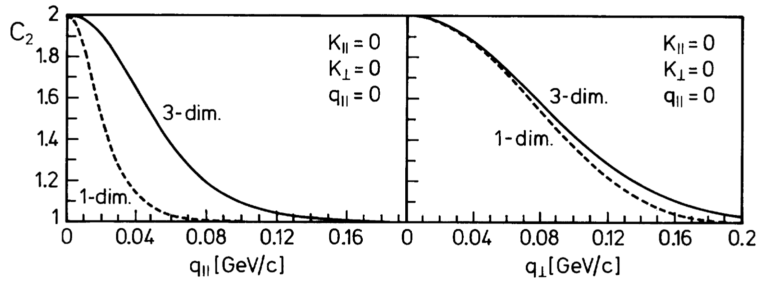

Fig. 10. Bose–Einstein correlation functions in longitudinal and transverse direction, for the three-dimensional (solid lines) and the one-dimensional calculations (dashed lines) (from Ref. [101]).

intuition about the role of transverse expansion in the determination of the transverse and longitudinal radius may be completely misleading. Only a systematic analysis based on (3 + 1)-D hydrodynamics clarified this issue.

In the present section we will discuss Bose–Einstein correlations of pions and kaons produced in nuclear collisions at SPS energies in the framework of relativistic hydrodynamics. Concrete applications were done for the symmetric reactions S + S and Pb + Pb at 200 AGeV. Many of the theoretical results were predictions at the time they were obtained. These predictions were subsequently confirmed in experiment. In [140] it was found that the transverse radius extracted from data on Bose–Einstein correlations (BEC) for O + Au at 200 AGev reached in the central rapidity region a value of about 8 fm. It was then natural to conjecture that this could be an indication of transverse flow [141–143]. In the meantime the experimental observation in itself has been qualified [144] and it now appears that the transverse radius obtained from the BEC data does not exceed a value of 4–5 fm (see however Ref. [143]). Motivated by this situation in Ref. [101] an investigation[55] of the role of three-dimensional hydrodynamical expansion on the space–time extension of the source was performed and compared with a (1 + 1)-D calculation.

Contrary to what one might have expected it was found that *transverse flow does not increase the transverse radius*. On the other hand, a strong dependence of the longitudinal radius on the transverse expansion was established.

Fig. 10 shows two typical examples of the Bose–Einstein correlations as functions of $q_{\|}$ and $q_{\perp}$ for the one- and the three-dimensional hydrodynamical solution. The dependence of $C_2(q_{\|})$ on transverse expansion agrees qualitatively with what one would expect. For a purely longitudinal expansion, the effective longitudinal radius of the source is larger than in the case of three-dimensional expansion, which is reflected in a decrease of the width of the correlation function (see also in Fig. 11 below).

On the other hand, the results for $C_2(q_{\perp})$ were at a first glance rather surprising. Naively one might have expected that the transverse flow would lead to an increase of the transverse radius, i.e., to a narrower correlation function $C_2(q_{\perp})$. However, in Fig. 10 the curves that describe the

[55] In Ref. [145] the dependence of the effective transverse radius on the transverse velocity field was investigated for a fixed freeze-out hypersurface. The effects of transverse expansion on the shape and position of the hypersurface are not considered there.

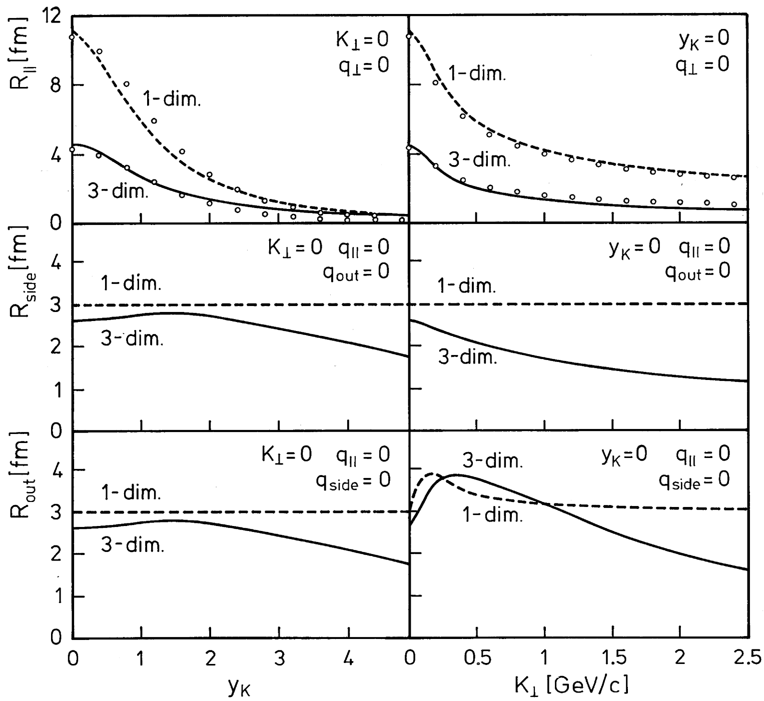

Fig. 11. Dependence of the longitudinal and transverse radii extracted from Bose–Einstein correlation functions on the rapidity y_K and average momentum $K_\perp$ of the pair. As before, solid lines correspond to the three-dimensional and dashed lines to the one-dimensional results. The open circles indicate values of $R_\parallel$ obtained from Eq. (5.4) (from Ref. [101]).

one- and the three-dimensional results are almost identical. If anything, one would conclude that the effective transverse radius is *smaller* in the presence of transverse expansion.

This effect can however be explained if one takes a closer look at the details of the hydrodynamic expansion process as investigated in Ref. [101]. Due to the correlation between the space–time point where a particle is emitted, and its energy–momentum, the effective radii obtained from Bose–Einstein correlation data present a characteristic dependence on the average momentum of the pair, K^μ. Fig. 11 shows the dependence of the effective radii $R_\parallel$, R_{side} and R_{out} on the rapidity y_K and the mean transverse momentum of the pair $K_\perp$, both for the one- and for the three-dimensional calculation. The longitudinal radius $R_\parallel$ becomes considerably smaller (by a factor of 2–3) if transverse expansion is taken into account. For the one-dimensional case, an approximate analytic expression has been derived for the y_K- and the $K_\perp$-dependence of the longitudinal "radii" in Refs. [146,147][56] (see also [95]):

$$R_\parallel = \sqrt{(2T_{\mathrm{f}}/m_\perp)}\tau_o/\cosh(y_K)\,, \tag{5.4}$$

[56] In this reference Eq. (5.4) is used to disprove the applicability of hydrodynamics to p–p reactions in the ISR energy range ($\sqrt{s} = 53\,\mathrm{GeV}$). Such a conclusion seems dangerous given the approximations involved both in the derivation of this formula as well as in the interpretation of the BEC measurements in the above reactions.

where $m_\perp = (m_{\perp 1} + m_{\perp 2})/2$ is the average transverse mass of the two particles, T_f is the freeze-out temperature and $\tau_o = (\partial u_\parallel/\partial x)^{-1}$ is the inverse gradient of the longitudinal component of the four-velocity in the centre (at $x = 0$).[57] In [101] one finds that this approximate expression describes $R_\parallel(K_\perp, y_K)$ for S + S reactions quite well, both for the one- and the three-dimensional case (see Fig. 11). However for Pb + Pb reactions the same formula fails to account for the data [148]. This is not surprising, because Eq. (5.4) is based on the assumption of boost invariance, i.e. no stopping. This assumption is not justified at SPS energies where there is considerable stopping. The inelasticity increases with atomic number and this may explain the breakdown of the above formula. This exemplifies the limitations of the boost-invariance assumption, an assumption which must not be taken for granted but in special circumstances.

In [149] it was proposed to use the information obtained from fitting the single inclusive distribution to constrain the parameters that enter into the hydrodynamic description, and then to calculate the transverse radius directly. Indeed, let Σ denote the hypersurface in Minkowski space on which hadrons are produced. Then one can define, e.g. a transverse radius

$$R_\perp = \frac{\int_\Sigma R\, p^\mu \mathrm{d}\sigma_\mu / (\exp[(p^\mu u_\mu - \mu)/T] - 1)}{\int_\Sigma p^\mu \mathrm{d}\sigma_\mu / (\exp[(p^\mu u_\mu - \mu)/T] - 1)}\,, \tag{5.5}$$

where u_μ, T and μ denote the four-velocity, temperature, and chemical potential on the hypersurface Σ, respectively as in Eq. (5.3).

It is interesting to note that this method for the determination of transverse radii based on the single inclusive cross sections provides a geometrical radius while the use of the second-order correlation function provides a coherence length (length of homogeneity).

Comparing the effective transverse radius $R_\perp$ extracted from the Bose–Einstein correlation function to the mean transverse radius as calculated directly in [149] according to Eq. (5.3), one finds in [101] that the two results agree to an accuracy of about 10%. This conclusion is confirmed and strengthened in a more recent study by Schlei [131] for kaon correlations.

Of course, this approach can be used only if a solution of the equations of hydrodynamics is available; with quasi-hydrodynamical methods this is not possible.

5.1.3. Role of resonances and coherence in the hydrodynamical approach to BEC

This problem was investigated using an exact (3 + 1)-D numerical solution of hydrodynamics in [150].

The source distribution $g(x, k)$ was determined from a three-dimensional solution of the relativistic hydrodynamic equations. Fig. 12 illustrates the effect of successively adding the contributions from ρ, ω, Δ and η decays to the BEC correlation functions of directly produced (thermal) π^- (dotted lines), in longitudinal and in transverse direction. The width of the correlation progressively decreases as the decays of resonances with longer lifetimes are taken into account, and the correlation loses its Gaussian shape. The long-lived η leads to a decrease of the intercept.

Pion versus kaon interferometry. Ideally, a comparison of pion and kaon interferometry should lead to conclusions concerning possible differences in the space–time regions where these particle

[57] Expression (5.4) for $R_\parallel$ denotes in fact the length of homogeneity L_h mentioned above. It refers to the region within which the variation of Wigner function is small. By definition $L_h \leq R$ where R is the geometrical radius.

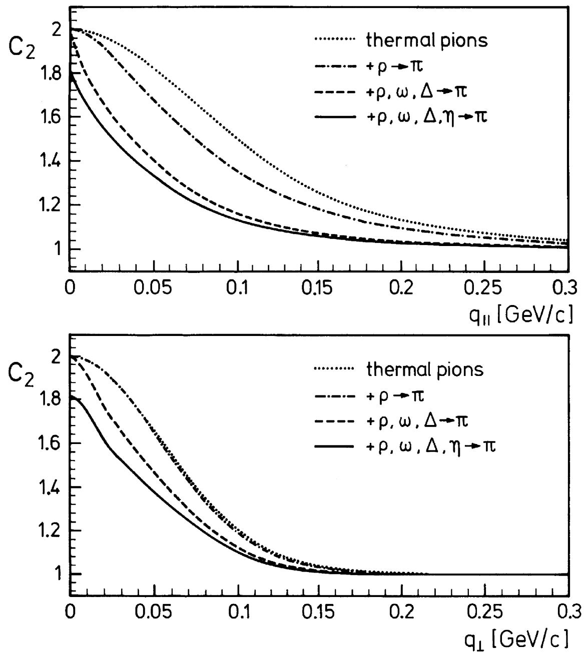

Fig. 12. Bose–Einstein correlation functions of negatively charged pions, in longitudinal and transverse direction. The separate contributions from resonances are successively added to the correlation function of direct (thermal) π^- (dotted line). The solid line describes the correlation function of all π^- (from Ref. [156]).

decouple from the hot and dense matter. It was proposed that kaons may decouple (freeze out) at earlier times and higher temperatures than pions [151]. Indeed, preliminary results had indicated that the effective longitudinal and transverse source radii extracted from $\pi\pi$ correlations were significantly larger than those obtained from KK correlations [152]. However, seen in Fig. 12, the BEC of pions are strongly distorted by the contributions from resonance decay. It was pointed out in Ref. [52] in a study based on the Lund string model that such distortions are not present for the BEC of kaons, and that consequently for the effective transverse radii one expects $R_\perp(K^\pm) < R_\perp(\pi^\pm)$, even in the absence of any difference in the freeze-out geometry of *directly produced* pions and kaons. These conclusions were confirmed in [156] within the hydrodynamical approach and one found furthermore that this effect is even more pronounced if one considers longitudinal rather than transverse radii. Furthermore, the interplay between coherence and resonance production which was not considered in [52] was studied in [150]. There are also some striking differences between [52,150] in the resonance production cross sections used.

In Fig. 13, BEC functions of π^- (solid lines) and of K^- (dashed lines) are compared, at $k_\perp = 0$ and $k_\perp = 1\,\mathrm{GeV}/c$, respectively. The dotted lines correspond to the BEC function of thermally produced π^-. It can be seen that the distortions due to the decay contributions from long-lived resonances disappear only at large $k_\perp$. Fig. 14 shows the effective radii $R_\parallel, R_{\mathrm{side}}$ and R_{out} as functions of rapidity and transverse momentum of the pair, both for $\pi^-\pi^-$ (solid lines) and for K^-K^- pairs (dashed lines). For comparison, the curves for thermally produced pions (dotted lines)

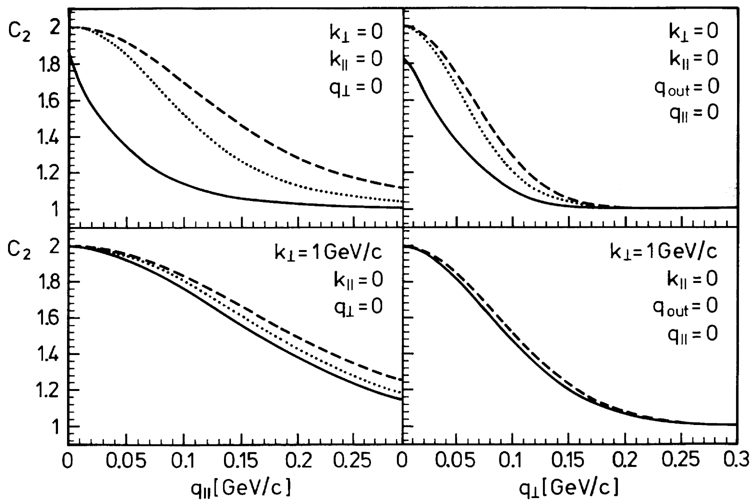

Fig. 13. Correlation functions of all π^- (solid lines), thermal π^- (dotted lines) and all K^- (dashed lines), for $k_\perp = 0$ and for $k_\perp = 1\,\mathrm{GeV}/c$ (from Ref. [56]).

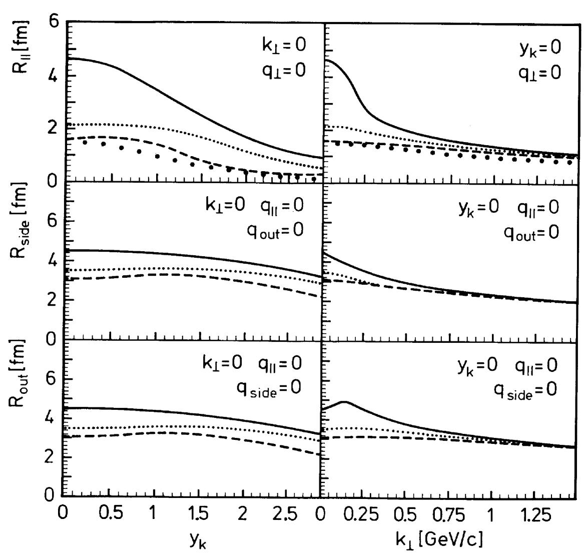

Fig. 14. Dependence of the longitudinal and transverse radii extracted from Bose–Einstein correlation functions on the rapidity y_k and average momentum $k_\perp$ of the pair, for all π^- (solid lines), thermal π^- (dotted lines) and all K^- (dashed lines). The full circles were obtained by substituting the value $\langle t_f(z=0, r_\perp)\rangle = 2\,\mathrm{fm}/c$ for the average lifetime of the system (calculated directly from hydrodynamics by averaging over the hypersurface) into Eq. (5.4), with $T_f = 0.139\,\mathrm{GeV}$ (from Ref. [56]).

are also included. The effective longitudinal radii extracted from $\pi^-\pi^-$ correlations are considerably larger than those obtained from K^-K^- correlations. In the central region the two values for $R_\parallel$ differ by a factor of ~ 2. For the transverse radii, the factor is ~ 1.3. A comparison between

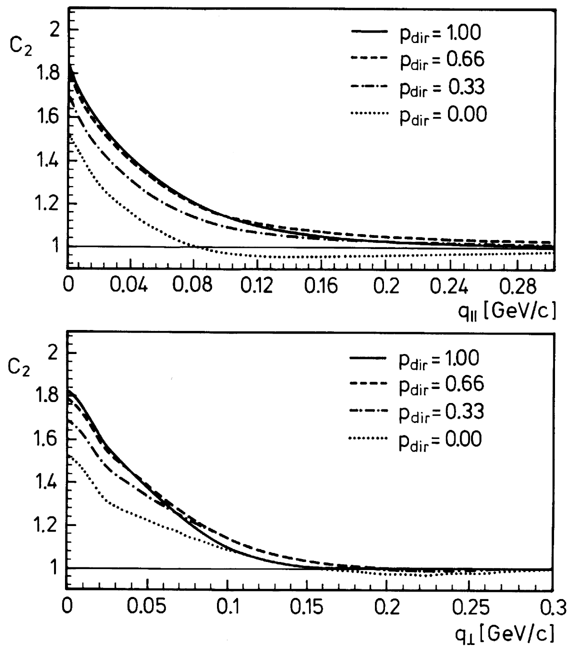

Fig. 15. $\pi^-\pi^-$ Bose–Einstein correlation functions in the presence of partial coherence (from Ref. [56]).

results for K^- and thermal π^- shows that part of this effect can be accounted for by kinematics (the pion mass being smaller than the kaon mass; see also Eq. (5.4)). Nevertheless, the large difference between the widths of pion and kaon correlation functions is mainly due to the fact that pion correlations are strongly affected by resonance decays, which is not the case for the kaon correlations. In the hydrodynamic scenario of Ref. [56], about 50% of the pions in the central rapidity region are the decay products of resonances [149], while less than 10% of the kaons are created in resonance decays ($K^* \to K\pi$ dominates, contributing with about 5%).

In Ref. [56] the problem of coherence within the hydrodynamical approach to BEC was also investigated. Fig. 15 shows the $\pi^-\pi^-$ correlation functions in the presence of partial coherence. In order to extract effective radii from Bose–Einstein correlation functions in the presence of partial coherence, Eq. (5.1) must be replaced by the more general form

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \lambda \cdot p_{\mathrm{eff}}^2 \exp\left[-\frac{1}{2}\sum(qR)^2\right] + \sqrt{\lambda}\cdot 2p_{\mathrm{eff}}(1-p_{\mathrm{eff}})\exp\left[-\frac{1}{4}\sum(qR)^2\right]. \quad (5.6)$$

5.1.4. Comparison with experimental data

Some of the predictions made in [56] for S + S reactions could be checked experimentally in Refs. [153,154], in particular the rapidity and transverse momentum dependence of radii and remarkable agreement was found. In [155] the hydrodynamical calculations were extended to Pb + Pb reactions and compared with S + S reactions and, where data were available, with experiment. The calculation of Bose–Einstein correlations (BEC) was performed using the

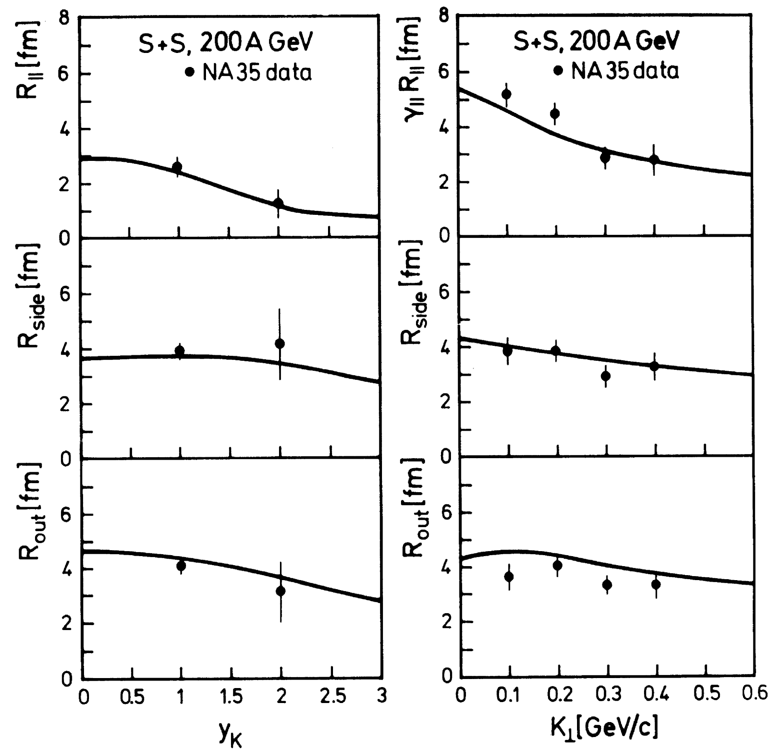

Fig. 16. Effective radii extracted from Bose–Einstein correlation functions as a function of the rapidity y_k of the pair and the transverse average momentum $K_\perp$ of the pair for all pions (from Ref. [155]).

formalism outlined in Ref. [150] including the decay of resonances. The hadron source was assumed to be fully chaotic.

Figs. 16 and 17 show the calculations for the effective radii $R_{\parallel}, R_{\mathrm{side}}$ and R_{out} as functions of rapidity y_K and transverse momentum $K_\perp$ of the pion pair compared to the corresponding NA35 and preliminary NA49 data [157,158], respectively. All these calculations, which in the case of S + S had been true predictions, agree surprisingly well with the data.[58] This suggests that our understanding of BEC in heavy ion reactions has made progress and confirms the usefulness of the Wigner approach when coupled with full-fledged hydrodynamics. An important issue in comparing data with theory is the detector acceptance of a given experiment. This is also discussed in detail in [155].

Another application of hydrodynamics to the QGP search in heavy ion reactions is due to Rischke and Gyulassy [159] who investigate the ratio $r = R_{\mathrm{out}}/R_{\mathrm{side}}$. Based on considerations due to Pratt [100], this quantity had been proposed by

[58] The EOS used in the hydrodynamical studies quoted above included a phase transition from QGP to hadronic matter. How critical this assumption is for the agreement with data is yet unclear and deserves a more detailed investigation. On the other hand, the very use of hydrodynamics is based on the assumption of local equilibrium, and this equilibrium is favoured by the large member of degrees of freedom due to a QGP.

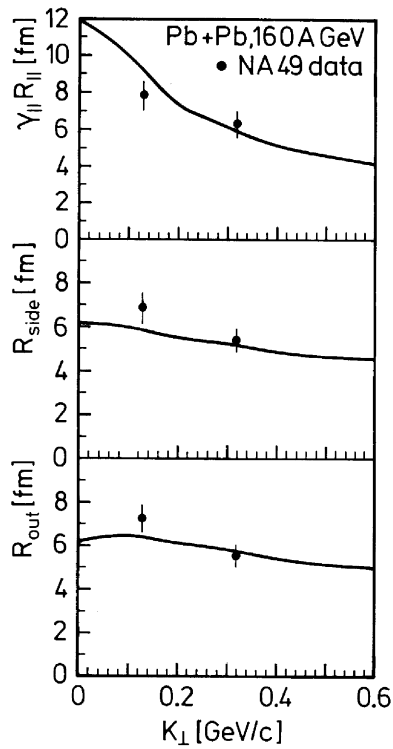

Fig. 17. Effective radii extracted from Bose–Einstein correlation functions as a function of the transverse average momentum $K_{\perp}$ of the pair for all pions compared with data (from Ref. [155]).

Bertsch and collaborators [160] as a signal of QGP. Under certain circumstances one could expect that for a long-lived QGP phase r should exceed unity, while for a hadronic system, due to final state interactions, the out and side sizes should be comparable.

The authors of [159] performed a quantitative hydrodynamical study of r in order to check whether this signal survives a more realistic investigation, albeit they did not take into account resonances. For directly produced pions it is found that r indeed reflects the lifetime of an intermediate (QGP) phase. However we have seen from [56,155], that for pion BEC, when resonances are considered, hydrodynamics with an EOS containing a long-lived QGP phase, leads (in agreement with experiment) to values of r of order unity.

To avoid this complication, in [161] it was proposed to consider kaons at large $k_{\perp}$. However even this proposal may have to be qualified, besides the fact that it will be very difficult to do kaon BEC at large $k_{\perp}$. Firstly, one has to recall that the entire formalism on which the r signal is based and in particular the parametrisation (5.1) is questionable. Secondly it remains to be proved that this signal survives if one imposes simultaneously the essential constraint due to the single inclusive distribution.[59] Furthermore, it is unclear up to what values of $k_{\perp}$ the Wigner formalism, (which is a particular case of the classical current formalism) on which the theory is based, is applicable. For these reasons the determination of the lifetime of the system via "pure" hydrodynamical considerations is certainly an alternative which deserves to be considered seriously, despite its own difficulties.

[59] I am indebted to B.R. Schlei for this remark.

5.1.5. Bose–Einstein correlations and quasi-hydrodynamics

As mentioned already, the initial motivation for proposing the Wigner function formalism for BEC was to explain why the experimentally observed second-order correlation functions C_2 were depending not only on the momentum difference $q = k_1 - k_2$ but also on their sum $K = k_1 + k_2$. However in the mean time it was shown [33,3] that this feature follows from the proper application of the space–time approach in the current formalism even without assuming expansion. Furthermore, the Wigner formalism is useful only at small q and cannot be applied in the case of strong correlations between positions and momenta while the current formalism is not limited by these constraints. As explained above the use of the Wigner formalism can be defended from the point of view of economy of thought if combined with bonafide hydrodynamics and an equation of state.

This notwithstanding, besides a few real, albeit numerical, hydrodynamical calculations, most phenomenological papers on BEC in heavy ion reactions (see e.g. [67,162,163,102,164–168, 68,69,169–178] have used the Wigner formalism without a proper hydrodynamical treatment, i.e. without solving the equations of hydrodynamics; hydrodynamical concepts like velocity and temperature were used just to parametrise the Wigner source function. While such a procedure may be acceptable as a theoretical exercise, it is certainly no substitute for a professional analysis of heavy ion reactions. This is a fortiori true when real data have to be interpreted.[60]

As exemplified in previous sections such a procedure is unsatisfactory, among other things because it can lead to wrong results.

The use of this "quasi-hydrodynamical" approach is even more surprising if one realises the fact that the Wigner formalism not only is not simpler than the more general current[61] formalism but it is also less economical. The number of independent parameters necessary to characterise the BEC within the Wigner formalism of Ref. [166] is[62] 10, i.e. it is as large as that in the current formalism. However the 10 parameters of [166] describe a very particular source,[63] as compared with that of the current formalism: besides the fact that the correlation function source is assumed to be Gaussian, it is completely chaotic and it can provide only the length of homogeneity L_h [67]. For the search of quark–gluon plasma, however, the geometrical radius R is relevant, because the energy density is defined in terms of R and the use of L_h instead of R leads to an overestimate of the

[60] A recent experimental paper [45] where such a procedure is used is a good illustration of the limits of quasi-hydrodynamical models. The analysis here concentrates on the resolution of the ambiguity between temperature and transverse expansion velocity of the source. It is clear that such an ambiguity is specific to quasi-hydrodynamics and is from the beginning absent in a correct hydrodynamical treatment. Moreover even this result may have to be qualified given some assumptions which underlay this analysis. To quote just two: (i) The assumption of boost invariance made in [45] decouples the longitudinal expansion from the transverse one. This not only affects the conclusions drawn in this analysis but prevents the (simultaneous) interpretation of the experimental rapidity distribution. (ii) The neglect of long-lived resonances which strongly influences the λ factor and thus also the extracted radii. Of course, despite the richness of the data, no attempt to relate the observations to an equation of state can be made within this approach.

[61] We remind that the classical current formalism, in opposition to the Wigner function formalism, does not apply only to small q and to semiclassical situations. Furthermore it allows for a correct treatment of new phenomena like particle–antiparticle correlations.

[62] For each value of $\boldsymbol{K}$ a Gaussian ellipsoid is described by three spatial extensions, one temporal extension, three components of the velocity in the local rest frame and the three Euler angles of orientation.

[63] To consider such an approach as "model independent" [165,179] is misleading.

energy density.[64] Furthermore, the physical significance of the parameters of the Wigner source is unclear if the Gaussian assumption does not hold.[65] Not only is there no a priori reason for a Gaussian form, but on the contrary, both in nuclear and particle physics as well as in quantum optics, there exists experimental evidence that in many cases an exponential function in $|q|$ is at small q a better approximation for C_2 than a Gaussian. Furthermore, in the presence of coherence, no single simple analytical function, and in particular no single Gaussian is expected to describe C_2. This is a straightforward consequence of quantum statistics.

Given the fact that good experimental BEC data are expensive both in terms of accelerator running time and man power, the use of inappropriate theoretical tools, when more appropriate ones are available, is a waste which has to be avoided. For the reasons quoted above we will not discuss in more detail the numerous and sometimes unnecessarily long papers which use quasi-hydrodynamical methods.

Concluding remarks. In a consistent treatment of single and double inclusive cross sections for identical pions via a realistic hydrodynamical model, resonances play a major role leading to an increase of effective radii of sources. Effective longitudinal radii are more sensitive to the presence of resonances than transverse ones.

From the hydrodynamical treatment we learn that the hadron source (the real fireball) is represented by a very complex freeze-out hypersurface (see Ref. [155]). The longitudinal and transverse extensions of the fireball change dynamically as a function of time, rather than show up in static effective radii. Thus, the interpretation of BEC measurements is also complicated.

When no quantitative comparison with data is intended, besides the current formalism, analytical approximations of the equations of hydrodynamics can be useful, because they allow a better qualitative understanding of hydrodynamical expansion. However, when a quantitative interpretation of experiments is intended and in particular a connection with the equation of state is looked for, the only recommendable method is full-fledged hydrodynamics.

5.1.6. Photon correlations and quasi-hydrodynamics

Photon correlations have been investigated within the context of quark–gluon plasma search, since they present certain methodological advantages as compared with hadrons. While experimentally genuine photon BEC in high-energy heavy ion reactions have not yet been unambiguously identified, because of the strong π^0 background and the small cross sections for photon production, there are several theoretical studies devoted to this topic. The advantage of photon BEC resides in the fact that, while correlations between hadrons are influenced by final state interactions, photon correlations are "clean" from this point of view. For high-energy physics, photons present another important advantage that they can provide direct information about the early stages of the interaction when quarks and gluons dominate and hadrons have not yet been created. In particular photon BEC contain information about the lifetime of the quark–gluon plasma [84–87,180,89]. Among other things it was argued, e.g. in [180] and confirmed in [89]

[64] One may argue that the "length of homogeneity" [67] L_h defined in terms of the Wigner function is a particular case of the correlation length L defined in terms of the correlator (see Section 4.3). While L_h is always limited from above by R, L can be either smaller or larger than R.

[65] Even if a Gaussian form would hold for directly produced pions, resonances would spoil it [56,155,131].

using a more correct formalism (see below), that the correlation function C_2 in the transverse direction can serve as a signal for QGP as it is sensitive to the existence of a mixed phase.

Unfortunately, some of these studies (Refs. [84–87,180]) besides being based on quasi-hydrodynamics, use an input formula for the second-order BEC, which is essentially incorrect (see Section 4.6). Besides this, some approximations made in [84], e.g. are inadequate. This question was analysed in more detail in [89] where it was found that only some of the results of Ref. [84] survive a more critical analysis.

The formula for the two-particle inclusive probability used in [84–87,180] reads

$$P_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \int d^4x_1 \int d^4x_2\, g(x_1,k_1)g(x_2,k_2)[1 + \cos((k_1 - k_2)(x_1 - x_2))] \,, \tag{5.7}$$

while Ref. [89] uses the more correct formula

$$P_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \int d^4x_1 \int d^4x_2\, g\left(x_1,\frac{k_1+k_2}{2}\right)g\left(x_2,\frac{k_1+k_2}{2}\right)[1 + \cos((k_1 - k_2)(x_1 - x_2))] \,. \tag{5.8}$$

(From Eq. (5.8) one gets for the second-order correlation function Eq. (4.58).) It is found in [89] that two rather surprising properties of the two-photon correlation function presented in [84] are artefacts of inappropriate approximations in the evaluation of space–time integrals. In [84], it has been claimed that the BEC function in the longitudinal direction (a) oscillates and (b) takes values below unity. As property (b) is inconsistent with general quantum statistical bounds, it was important to clarify the origin of this discrepancy. On the other hand, it has been confirmed in [89] that the correlation function in the transverse direction does exhibit oscillatory behaviour in the out component of the momentum difference. Furthermore, in this reference a change of the BEC function in Δy from Gaussian to a two-component shape with decreasing transverse photon momentum was found which may serve as evidence for the presence of a mixed phase and, hence, as a QGP signature.

However even after correcting the wrong input BEC formula it is questionable whether the other approximations made in Refs. [84–87,180,89] may not invalidate the above result. Besides the use of a simplified hydrodynamical solution one has to recall that (i) the Wigner formalism like the more general classical current formalism, is limited to small momenta k of produced particles (no recoil approximation), (ii) besides this general limitation to small k the Wigner formalism is useful only for small differences of momenta q and for weak correlations between coordinates and momenta, and (iii) although Eq. (4.58) does not suffer from the violation of unitarity disease mentioned in Section 4.6 of Chapter 3, it is based on an approximation which makes it sometimes inapplicable for photons (see Section 4.5).

From the above discussion one may conclude that the experimental problems of photon BEC are matched by theoretical problems yet to be solved.

5.2. Pion condensates

One of the most interesting phenomena related to Bose–Einstein correlations is the effect of Bose condensates. The remarkable thing about this effect is that it is not specific for particle or nuclear physics, but occurs in various other chapters of physics like superconductivity and superfluidity. Moreover, recently not only the condensation of a gas of atoms has been experimentally achieved [185] but the quantum statistical coherence of these systems has been experimentally proven through Bose–Einstein correlations [24]. The proposal to use BEC for the detection of condensates was made a long time ago [59]. The more recent developments in heavy ion reactions have made this subject of current interest. Already in experiments at the SPS (e.g., Pb + Pb at $E_{\mathrm{beam}} = 160$ AGeV) secondary particles are formed at high number densities in rapidity space [181] and in future experiments at RHIC and the LHC one expects to obtain even higher multiplicities of the order of a few thousand particles per unit rapidity. If local thermal (but not chemical) equilibrium is established and the number densities are sufficiently large, the pions may accumulate in their ground state and a Bose condensate may be formed.[66] A specific scenario for the formation of a Bose condensate, namely, the decay of short-lived resonances, was discussed in Ref. [183] where conditions necessary for the formation of a Bose condensate in a heavy ion collision were investigated. In Ref. [183] it was found that if a pionic Bose condensate is formed at any stage of the collision, it can be expected to survive until pions decouple from the dense matter, and thus it can affect the spectra and correlations of final state pions.

In Ref. [184] one investigated the influence of such a condensate on the single inclusive cross section and on the second-order correlation function of identically charged pions (Bose–Einstein correlations BEC) in hadronic reactions for *expanding* sources. A hydrodynamical approach was used based on the HYLANDER routine. The Bose–condensate affects the single inclusive momentum distributions $E\mathrm{d}N/\mathrm{d}^3k$, the momentum-dependent chaoticities p and the Bose–Einstein correlation functions C_2 only over a limited momentum range. This is due to the fact that in a condensate there exists a maximum velocity (which implies also a maximum momentum difference q_{max}) and leads to a very characteristic structure in single and double inclusive spectra.

In Fig. 18 the results of the numerical evaluations of the single inclusive momentum distributions $E(\mathrm{d}N/\mathrm{d}^3k)$, the momentum-dependent chaoticities p and the Bose–Einstein correlation functions C_2 are shown for a spherically and for a longitudinally expanding source. The presence of a Bose–condensate of only 1% results in a decrease of the intercept by about 15%. Furthermore due to a limited value of q_{max} a part of the tail of the two-particle correlation functions is not affected by the pionic Bose-condensate and a peak appears. To what extent such peaks can be observed in experimental data depends among other things on the size of the source, the details of the freeze-out, the width of the momentum distribution in the bosonic ground state, and detector acceptance.

Plasma droplets? If the phase transition from hadronic matter to QGP and vice versa is of first order then one could expect the formation of a mixed phase, in which QGP and hadronic matter coexist. Such a mixed phase manifests itself in the hydrodynamical evolution of the system [186] and it influences among other things the transverse momentum distribution of photons as seen in Section 5.1.6. It was suggested by Seibert [187] that the mixed phase could also lead to a granular structure which might be seen in the fluctuations of the velocity distributions of secondaries

[66] Recently a different type of pion condensate, the disordered chiral condensate, has received much attention in the literature. It has been argued [182] that such a condensate would lead to the creation of squeezed states.

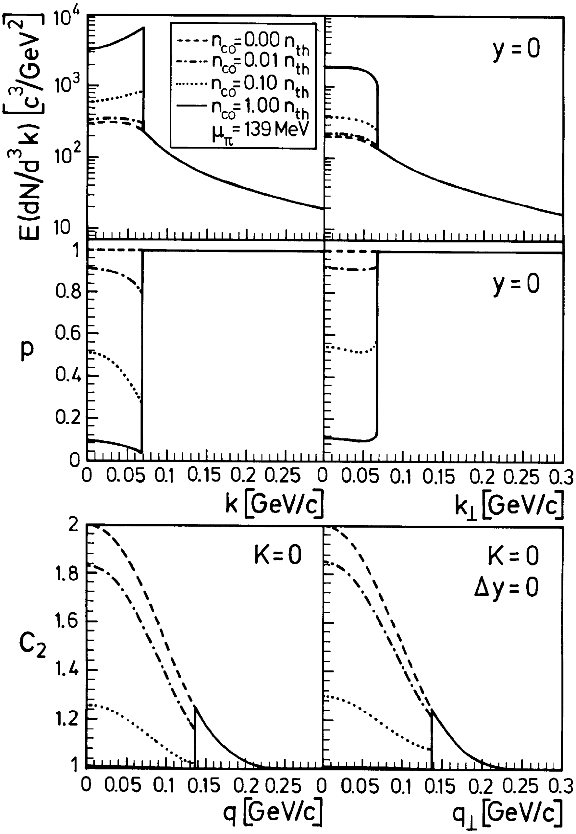

Fig. 18. Single inclusive spectra, momentum dependent chaoticities and two-particle BE correlation functions for a spherically and a longitudinally expanding source. The different line styles correspond to different condensate densities n_{co} compared to the thermal number densities n_{th}. μ_π is the pionic chemical potential (from Ref. [184]).

produced in the hadronisation stage. Pratt et al. [188] (see also [189] proposed subsequently that in Bose–Einstein correlations, too, one might see a signature of this granularity.

6. Correlations and multiplicity distributions

6.1. From correlations to multiplicity distributions

An important physical observable in multiparticle production is the multiplicity distribution[67] $P(n)$, i.e., the probability to produce in a given event n particles. The link between the multiplicity

[67] The multiplicity distribution will be denoted sometimes in the following also by MD.

distribution $P(n)$ and BEC is represented by the density matrix ρ, since it is the same ρ which appears in the definition of $P(n)$:

$$P(n) \equiv \langle n|\rho|n\rangle \tag{6.1}$$

and the definition of correlation functions (see e.g. Eq. (4.4)). Usually, one expresses ρ in terms of the $\mathscr{P}(\alpha)$ representation (see Section 2.2) and by using for $\mathscr{P}(\alpha)$ simple analytical expressions one is able to derive the most characteristic forms of $P(n)$ in an analytical form, like the Poisson or the negative binomial representation (see e.g. [5]).

There exist however physically interesting cases where no analytical expression for the multiplicity distribution $P(n)$ exists, but instead the moments of P are given.

From the phenomenological point of view the approach to MD via moments presents sometimes important advantages because it allows the construction of an *effective* density matrix from the knowledge of a few physical quantities like the correlation lengths and mean multiplicity, which in turn can be obtained from experiment (see e.g. [190]). In the following we will address this aspect of the problem, especially since in this way the link between correlations and MD becomes clearer.

We start by recalling some definitions. Besides the normal moments of the MD given by

$$\langle n^q\rangle \equiv \sum_n P(n)n^q , \tag{6.2}$$

one uses frequently the factorial moments

$$\Phi_q \equiv \sum_n P(n)(n(n-1)\cdot \ldots \cdot(n-q+1)) . \tag{6.3}$$

These can be expressed in terms of the inclusive correlation functions ρ_q through the relation

$$\Phi_q = \int_\Omega \mathrm{d}\omega_1 \ldots \int_\Omega \mathrm{d}\omega_q\, \rho_q(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_q) = \left\langle \frac{n!}{(n-q)!} \right\rangle . \tag{6.4}$$

Eqs. (6.3) and (6.4) illustrate the fundamental fact that the inclusive cross sections ρ_q and thus the correlation functions determine the moments of the MD, which are nothing but the integrals of ρ_q. Although this relation between moments of MD and correlation functions is a straightforward aspect of multiparticle dynamics, the connection between MD and correlations has often been overlooked. This is in part due to the fact that measurements of correlations are, for reasons of statistics and other technical considerations, frequently performed in different (narrower) regions of phase space than measurements of MD.[68] However the importance of the use of the relationship between MD and correlations can hardly be overemphasised, just because of the different experimental methods used in the investigation of these two observables. In the absence of a theory of

[68] This is, e.g. the case where MD are measured with no proper identification of the particles, while BEC refer of course to identical particles. Thus the UA5 experiment [191], which discovered the violation of KNO scaling in MDs, measured only *charged* particles, without distinguishing between positive and negative charges. However, the rapidity region accessible in this experiment was much broader than the corresponding region in the UA-1 experiment [192] where correlation measurements were performed and where a distinction between positive and negative charges could be made.

multiparticle production, the form of the correlators and the amount of chaoticity are unknown and have therefore to be parametrised and then determined experimentally in correlation experiments. Both the parametrisation and the measurements are affected by errors. Similar considerations apply for MD, but because of the different experimental conditions under which correlation measurements and MD measurements are made, the corresponding errors are different. (See also the discussion of the importance of higher-order correlation, Section 2.2.1.) Therefore from a phenomenological and practical point of view, MD and correlations are rather complimentary and have to be interpreted together. In the following we will exemplify the usefulness of this point of view.

6.1.1. Rapidity dependence of MD in the stationary case

The dependence of moments of multiplicity distributions $P(n)$ on the width of the bins in momentum or rapidity space has been in the centre of multiparticle production studies for the last 15 years. It got much attention after: (1) the experimental observation [191] by the UA5 collaboration that the normalised moments of $P(n)$ in the rapidity plateau region increase with the width of the rapidity window Δy; (2) the proposal by Bialas and Peschanski [193] that this behaviour, which at a first look was power like may reflect "intermittency", i.e. the absence of a fixed scale in the problem, which could imply that self-similar phenomena play a role in multiparticle production.

However soon after this proposal was published it was pointed out in [194] that the quantum statistical approach, presented in Section 2.2 and which implies a *fixed* scale,[69] predicts a similar functional relationship between the moments of MD and Δy. In Fig. 19 from [194] some examples of this behaviour are plotted and compared with experimental data. For small $-\delta y$ the (semilogarithmic) plot can be approximated by a power function as indicated by the data. Recalling that the QS formalism applies to identical particles, it follows that BEC could be at the origin of the so-called intermittency effect. This point of view was corroborated subsequently also in [195,196] and was confirmed experimentally by the observation that the "intermittency" effect is strongly enhanced when identical particles are considered[70] and/or when studied in more than one dimension.[71] For a more recent very clear confirmation of this point of view refer to the studies by Tannenbaum [197]. Further developments related to "intermittency" will be discussed in Section 6.3.

6.1.2. Rapidity dependence of MD in the non-stationary case

The assumption of translational invariance in rapidity permitted to apply the quantum optical formalism, in which time has the analogous property, to MD and led to a simple interpretation of the observed broadening of the MD with the decrease of the width of the rapidity window in high-energy reactions. However stationarity in rapidity is expected to hold only in the central region (and only at high energies). Indeed experimental data on proton(antiproton)–proton

[69] This scale is the correlation length ξ of Eqs. (2.24) or (2.25).

[70] The UA5 data refer to a mixture of equal numbers of positive and negative particles; this dilutes the BEC effect.

[71] This last observation also supports the idea that BEC is the determining factor in intermittency because the integration over transverse momentum implied by a one-dimensional y investigation diminishes the BEC.

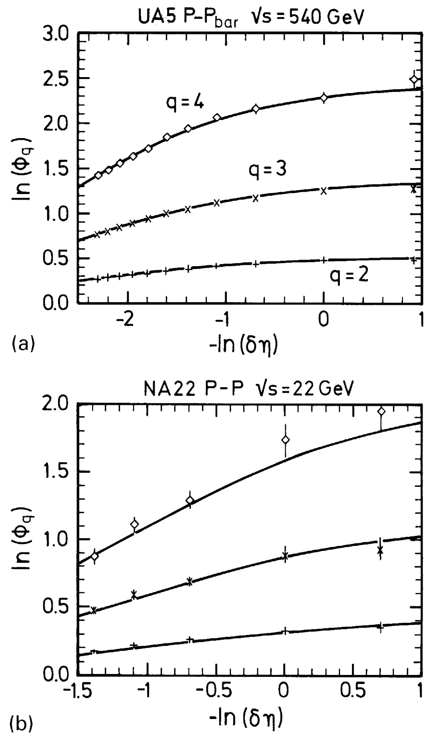

Fig. 19. Normalised factorial moments Φ_q of order q in finite (pseudo)rapidity windows of width δy around $y_{\mathrm{CMS}} = 0$, plotted against $\delta\eta$ (from Ref. [194]).

collisions in the energy range $52 < \sqrt{s} < 540\,\mathrm{GeV}$ show that if one considers shifted rapidity bins along the rapidity axis the MD in these bins depend on the position of the bin: in the central region the MD is broader and can be described by a negative binomial distribution while in the fragmentation region it is narrower and can be described by a Poisson MD [198]. The mean multiplicity in the central region increases faster (approximately like $s^{1/4}$) with the energy than in the fragmentation region. This was interpreted in [198] as possible evidence for the existence of two sources, one of chaotic nature localised in the central region and another coherent in the fragmentation region. This interpretation is in line with the folklore that gluons which interact stronger (than quarks) form a central blob which may be equilibrated, while the fragmentation region is populated by throughgoing quarks associated with the leading particles.[72]

A few years later the NA35-collaboration [200] measured BEC in ^{16}O–Au reactions at 200 GeV/nucleon in a relatively broad y region and found evidence for a larger and more chaotic source in the central rapidity region, and a smaller and more coherent source in the fragmentation region. It was then natural to correlate (see Ref. [28]) the two observations, i.e. that referring to MD

[72] For a microscopic interpretation of this effect in terms of a partonic stochastic model, see Ref. [199].

and that referring to BEC. Taken together the credibility of the conjecture made in Ref. [198] is strongly enhanced, just because we face here different physical observables and different experiments, each with its own specific corrections and biases. Moreover it was pointed out in [28] that experiment [200] did not necessarily imply that the two sources were independent, but could also be interpreted as due to a single partially coherent source.

Indeed consider in a simplified approach as used, e.g. in quantum optics a superposition π of two fields one coherent denoted by π_c and another chaotic denoted by π_{ch} so that $\pi = \pi_c(k_\perp^{(1)}) + \pi_{ch}(k_\perp^{(2)})$ and $Q_\perp = k_\perp^{(1)} - k_\perp^{(2)}$. Assuming boost invariance the correlator depends only on $k_\perp$ and we have for the second-order correlation function

$$C_2 = 1 + 2p(1-p)\mathrm{e}^{-Q_\perp^2 R_\perp^2/2} + p^2\mathrm{e}^{-Q_\perp^2 R_\perp^2} , \tag{6.5}$$

where the transverse "radius" $R_\perp$ plays the role of the correlation length and

$$p = \langle|\pi_{ch}\rangle|^2\rangle/(|\pi_c|^2 + \langle|\pi_{ch}\rangle|^2\rangle) .$$

Assume now that the chaoticity p is rapidity dependent so that in one rapidity region, denoted by (A), $p(A) \approx 1$. In that region then the third term in Eq. (6.5) dominates, i.e.

$$C_2 \approx 1 + p^2(A)\mathrm{e}^{-Q_\perp^2 R_\perp^2(A)} . \tag{6.6}$$

In the parametrisation used in [200], according to which we have two independent sources, this suggests that the effective radius of source A is $R_\perp(A)$. Conversely, for the more coherent region denoted by B, $p(B)$ is small and C_2 reads

$$C_2 \approx 1 + 2p(B)(1-p(B))\mathrm{e}^{-Q_\perp^2 R_\perp^2(B)/2} \tag{6.7}$$

with an effective radius $R_\perp(B)/\sqrt{2} < R_\perp(A)$.

This corresponds qualitatively to the observations made in Ref. [200]. Unfortunately, these observations have not yet been confirmed by another, independent experiment so that the reader should view these considerations with prudence.[73] In any case they prove the usefulness of a global analysis which incorporates both BEC and MD. On the other hand, a dedicated simultaneous investigation of the rapidity dependence of these two observables appears very desirable.

6.1.3. *Energy dependence of MD and its implications for BEC; long-range fluctuations in BEC and MD*

This subject has been discussed recently in [202]. Besides the rapidity dependence, the dependence of MD on the centre of mass energy of the collision, $\sqrt{s}$, constitutes an important topic in the study of high-energy multiparticle production processes. This energy dependence is usually discussed in terms of the violation [191] of KNO scaling [203]. KNO scaling implies that the normalised moments $\langle n^m\rangle/\langle n\rangle^m$ are constant as a function of s (for high energies, i.e. large $\langle n\rangle$, these moments coincide with the normalised factorial moments). For charged particles it turned out that while KNO scaling is approximately satisfied over the range of ISR energies ($20\,\mathrm{GeV} \le \sqrt{s} \le 60\,\mathrm{GeV}$), it is violated if one goes to SPS-Collider energies ($200\,\mathrm{GeV} \le \sqrt{s} \le 900\,\mathrm{GeV}$), i.e. one finds a considerable increase of multiplicity fluctuations with increasing

[73] For various caveats concerning the analysis of BEC and MD data see Refs. [200,191,198,28,60,201].

energy. In the following, we will show how BEC can be used to understand the origin of this s-dependence. To do this, one needs to distinguish between long-range (dynamical) correlations (LRC) and short-range correlations (SRC).[74] For identical bosons, one important type of SRC are BEC which reflect quantum statistical interference. In addition, there exist dynamical SRC like final state interactions, which however are quite difficult to be separated from BEC. Up to 1994 one usually assumed that LRC do not play an important part in BEC measurements (see however [195,196]) in the sense that for $Q > 1\,\mathrm{GeV}$ the two-particle correlation functions do not significantly exceed unity.

However in Ref. [202] evidence, based on observational data, was presented showing that this is not the case and that new and important information about LRC is contained in the BEC data obtained by the UA1-Minimum-Bias Collaboration [192]. In principle, the observed increase of multiplicity fluctuations with $\sqrt{s}$ could be due to a change of the SRC as seen in BEC, i.e. of the chaoticity and radii/lifetimes. This possibility was discussed in [207] but could not be tested because of the lack of identical particle data for multiplicity distributions at Collider energies. At that time only the UA5 data [191] for multiplicity distributions of charged particles were available. Furthermore, up to this point, the effect of LRC had only been studied in terms of two-particle correlations as a function of rapidity difference, i.e. in one dimension. With the advent of the newly analysed UA1 data [192] for identical particles in three dimensions (essentially in $Q^2 = -(k_1 - k_2)^2$) this situation changed. The analysis of [202] led to the conclusion of the existence in BEC data of long-range fluctuations in the momentum space density of secondaries and to the realisation that the increase with energy of multiplicity fluctuations is to a great extent due to an increase of the asymptotic values of the m-particle correlation functions $C_m^{\mathrm{asympt.}}$, i.e. their values in the limit of large momentum differences where BEC do not play a role.[75]

In Ref. [192], the UA1 collaboration presents the two-particle correlation of negatively charged secondaries as a function of the invariant momentum difference squared $Q^2 = (\boldsymbol{k}_1 - \boldsymbol{k}_2)^2 - (E_1 - E_2)^2$. The data (see Fig. 20) have two unusual features: (I) at large Q^2 the correlation function saturates above unity, and (II) at small Q^2 it takes on values above 2. The higher-order correlation functions also exhibit property (I) [192].[76] By comparing the asymptotic values of the correlation functions at large momentum differences $C_m^{\mathrm{asympt.}}$ ($m = 2, \ldots, 5$), with the normalised factorial moments, $\phi_m \equiv \langle n(n-1)\cdot \ldots \cdot(n-m+1)\rangle/\langle n\rangle^m$ in the momentum space region $|y| \le 3$, $k_\perp > 0.15\,\mathrm{GeV}$ one finds that the contribution of the BE interference peak to the moments is negligible for such large rapidity windows. Herefrom one concludes in [202] that (I) indicates the presence of LRC in the momentum space density of secondary particles and that it is quite plausible (see below) that (II) has to a great extent the same explanation. We sketch here the arguments of Ref. [202].

In general, LRC may be related to fluctuations in impact parameter or inelasticity, or fluctuations in the number of sources. In what follows, let us label these fluctuations by a parameter α.

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[74] The importance of making this distinction was pointed out, among other things, in [204,205]. In [206] rapidity correlations were measured for events at fixed multiplicity in order to get rid of the effect of LRC.

[75] The effect of LRC on BEC was discussed in Ref. [208], where several models specific for nucleus–nucleus collisions were considered, but at that time no evidence for this effect could be found.

[76] For higher-order correlations the equivalent of property (II) is $C_m > m!$ (see also below). The values of $C_m(\boldsymbol{k}, \ldots, \boldsymbol{k})$ for $m > 2$ are apparently not yet available.

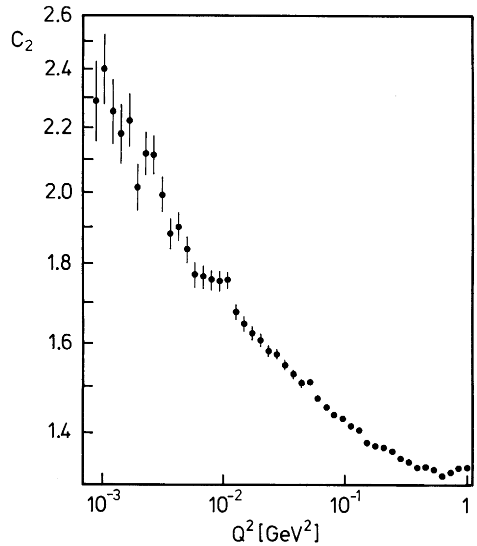

Fig. 20. Second-order correlation function for negative particles at $\sqrt{s} = 630\,\mathrm{GeV}$ (from Ref. [202]).

The m-particle Bose–Einstein correlation function at a fixed value of α is given by

$$C_m(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m|\alpha) = \rho_m(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m|\alpha)/\rho_1(\boldsymbol{k}_1|\alpha) \cdot \ \dots \ \cdot \rho_1(\boldsymbol{k}_m|\alpha) \ , \tag{6.8}$$

where $\rho_\ell(\boldsymbol{k}_1, \dots, \boldsymbol{k}_\ell|\alpha)$ are the ℓ-particle inclusive distributions.

The fluctuations in α are described by a probability distribution $h(\alpha)$ with

$$\int \mathrm{d}\alpha \, h(\alpha) = 1 \ . \tag{6.9}$$

If the experiment does not select events at fixed α, the measured inclusive distributions are

$$\rho_m(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m) = \langle \rho_m(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m|\alpha) \rangle \ , \tag{6.10}$$

where the symbol $\langle \dots \rangle$ denotes an average over the fluctuating parameter α, i.e.

$$\langle X(\alpha) \rangle \equiv \int \mathrm{d}\alpha \, h(\alpha) X(\alpha) \tag{6.11}$$

The m-particle correlation function at the intercept reads

$$C_m(\boldsymbol{k}, \dots, \boldsymbol{k}) = m! \, \langle \alpha^m \rangle / \langle \alpha \rangle^m \ , \tag{6.12}$$

where the symbols $\langle \ \rangle$ refer to averaging with respect to $h(\alpha)$. At large momentum differences one has

$$C_m(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m) \to \langle \alpha^m \rangle / \langle \alpha \rangle^m = C_m^{\text{asympt.}} \quad \text{for } |\boldsymbol{k}_i - \boldsymbol{k}_j| \to \infty \quad (i \neq j, \ i,j = 1, \dots, m) \ , \tag{6.13}$$

i.e., the m-particle correlation functions can have intercepts above $m!$ and saturate at values above unity for large momentum differences.

The most obvious candidate for the function $h(\alpha)$ is the inelasticity distribution which describes the event-to-event fluctuations of the inelasticity K. With the identification $\alpha \equiv \langle n(K)\rangle$, where $\langle n(K)\rangle$ is the mean multiplicity at inelasticity K, one obtains from the above considerations a first "experimental" information about this important physical quantity at Collider energies. Previous experimental information about this distribution was derived in [209] from the data of Ref. [210] at $\sqrt{s} \simeq 17\,\mathrm{GeV}$.

The conclusions obtained in [202] about LRC are based among other things on the different normalisations used in different experiments. Some tests related to these conclusions are proposed:

- Analysis of BEC at lower energies (NA22 range) as well as at $\sqrt{s} = 1800\,\mathrm{GeV}$ with the same normalisation as that used by the UA1 Collaboration. The values of C_2 at $Q^2 > 1\,\mathrm{GeV}^2$ and possibly also at very small Q^2 obtained in this way should exceed those obtained with the fixed multiplicity normalisation in the same experiments. The following inequalities for $C_2(Q^2 > 1\,(\mathrm{GeV})^2)$, should be observed if this normalisation is used:

  $$C_2(\mathrm{NA22}) < C_2(\mathrm{UA1}) < C_2(\mathrm{Tevatron}) .$$

- Analysis of BEC at UA1 energies with the same normalisation as that used so far by the NA22 and Tevatron groups (fixed multiplicity). The enhancement of C_2 at large Q (and possibly also at small Q) observed so far, should disappear to a great extent.

6.2. Multiplicity dependence of Bose–Einstein correlations

The operators for the field (intensity) and number of particles do not commute. This means that measurements of "ideal" BEC can be performed only when no restriction on the multiplicity n, which fluctuates from event to event, is made. In practice, however, very often such restrictions are imposed, either because of technical reasons or because of theoretical prejudices. To the last category belong considerations imposed by the search for QGP in high-energy heavy ion reactions. Thus one expects that by selecting events with $n \geq n_{\min}$, where $n_{\min}$ is in general an energy-dependent quantity, one gets information about the interesting "central" collisions.

Another reason why multiplicity constraints are of practical importance for QGP experiments is the need to compare various QGP signals in a given event and at the same time determine for that event the radius, lifetime and chaoticity of the source, among other things in order to be able to estimate the energy density achieved in that event. This means that for QGP search it is interesting to perform interferometry measurements for single events, which of course have a given multiplicity.

For these reasons the investigation of the multiplicity dependence of BEC constitutes an important enterprise which we shall address in the following section.

6.2.1. The quantum statistical formalism

Correlation functions defined in quantum statistics and used in quantum optics refer to ensemble averages of intensities[77] I of fields π, where

$$I(\boldsymbol{k}_\perp, y) = |\pi(\boldsymbol{k}_\perp, y)|^2 . \tag{6.14}$$

[77] Herefrom the name "intensity interferometry".

The total multiplicity n of *identical particles* over a given phase space region is given by

$$n = \int \mathrm{d}\boldsymbol{k}_\perp^2 \int \mathrm{d}y |\pi(\boldsymbol{k}_\perp, y)|^2 . \tag{6.15}$$

Both the field $\pi(\boldsymbol{k}_\perp, y)$ and the intensity $I(\boldsymbol{k}_\perp, y)$ are stochastic variables. Averaging over an appropriate ensemble, we get the mean total multiplicity

$$\langle n \rangle = \int \mathrm{d}\boldsymbol{k}_\perp^2 \int \mathrm{d}y \langle |\pi(\boldsymbol{k}_\perp, y)|^2 \rangle . \tag{6.16}$$

In [211] the dependence of BEC within the QS formalism on the total multiplicity n and on n_{min} was investigated and it was found that the size of this effect is (especially at low $\langle n \rangle$) surprisingly large and must not be ignored, as had been done before. Both the n_{min} constraint and the $\langle n \rangle$ constraint lead to a decrease of the correlation function C_2 at fixed $y_g = y_1 - y_2$, i.e. to an antibunching effect. The last effect can be approximated, except for very large n and small y_g by a simple analytical formula

$$C_{(2)}^{(n)}(y_g) \approx C_{(2)}(y_g)(n-1)/nf_2 , \tag{6.17}$$

where C_2^n denotes the correlation function at fixed n and f_2 is the reduced factorial moment.

These results show that BEC parameters like radii, lifetimes, and chaoticity do depend on the particular experimental conditions under which the measurements are performed.

It is worth mentioning that a multiplicity dependence of BEC was observed experimentally for p–p and α–α reactions at $E_{cm} = 53$ and 31 GeV, respectively already in [212]. There it was found that the transverse radius increases with the multiplicity of charged particles n_{charged}. This effect was interpreted by Barshay [213] to be a consequence of the impact parameter dependence. The same effect was seen in heavy ion reactions [200] and got a similar interpretation in [176].[78] The interpretation in terms of impact parameter dependence could be checked directly in heavy ion reactions since here the impact parameter can be determined on an event-by-event basis.

Another mechanism for the increase of radius with multiplicity was proposed by Ryskin [214]. He pointed out that in high multiplicity events, which many authors associate with large transverse momenta of partons and thus with a regime where perturbative QCD applies, one expects that the size of the hadronization region should increase with the multiplicity like $\sqrt{n}$.

A related topic is the dependence of BEC on the rapidity density $d = \Delta n / \Delta y$, which has also been observed experimentally in $\bar{p}$–p reactions at the CERN SPS Collider [215] and at the Fermilab tevatron [216]. Using a parametrisation

$$C_2 = 1 + \lambda \exp(-R^2 Q^2) , \tag{6.18}$$

where Q is the invariant momentum transfer, it was found that R increased with d while λ decreased with d. This last observation is compatible with the results from [211]; a more quantitative comparison would be possible only if, among other things, the data were parametrised in a way more consistent with quantum statistics. Thus the four momentum difference Q is not an ideal

[78] The approach of [176] combines simplified (1-D) hydrodynamics with a multiple scattering model, which also exploits the impact parameter dependence.

variable for BEC (see Section 2.1.5) and the coherence effect has to be taken into account as outlined in Section 2.2 and not by the simple empirical λ factor.

6.2.2. The wave-function formalism; "*pasers*"?

The dependence on total multiplicity of BEC was investigated also within the wave-function formalism. In Section 2.1 where the GGLP theory was presented it was pointed out that the wave-function formalism may be useful for exclusive processes or for event generators. Indeed, in a first approximation, the wave function ψ_n of a system of n identical bosons, e.g. can be obtained from the product of n single-particle wave functions ψ_1 by symmetrisation. Then the calculation of $C_{(2)}^{(n)}$ is in principle straightforward and follows the lines of GGLP.

However, when the multiplicities n become large (say $n > 20$) the explicit symmetrisation of the wave-function formalism becomes difficult. This leads Zajc [217] to use numerical Monte Carlo techniques for estimating n particle symmetrised probabilities, which he then applied to calculate two-particle BEC. He was thus able to study the question of the dependence of BEC parameters on the multiplicity. For an application of this approach to Bevalac heavy ion reactions, see [218]. Using as input a second-order BEC function parametrised in the form (6.18) in Ref. [217] it was found (and we have seen above that this was confirmed in [211],) that the "incoherence" parameter λ decreased with increasing n.[79]

This of course does not mean that events with higher pion multiplicities are denser and more coherent. On the contrary, Ref. [217] concluded that the above results have to be used in order to eliminate the *bias* introduced by this effect into experimental observations.[80]

The authors of [219,220] however did not share this opinion. Ref. [219] went even so far as to deduce the possible existence of pionic lasers from considerations of this type.

Ref. [219] starts by proposing an algorithm for symmetrising the wave functions which presents the advantages that it reduces the computing time very much when using numerical techniques, which is applicable also for Wigner-type source functions and not only plane wave functions, and which for Gaussian sources provides even analytical results. Subsequently in Ref. [221] wave packets were symmetrised and in special cases the matrix density at fixed and arbitrary n was derived in analytical form. This algorithm was then applied to calculate the influence of symmetrisation on BEC and multiplicity distributions. As in [217] it is found in [219] that the symmetrisation produces an effective decrease of the radius of the source, a broadening of the multiplicity distribution $P(n)$ and an increase of the mean multiplicity as compared to the non-symmetrised case. What is new in [219] is (besides the algorithm) mainly the meaning the author attributes to these results.

In a concrete example Ref. [219] considers a non-relativistic source distribution S in the absence of symmetrisation effects:

$$S(k,x) = [1/(2\pi R^2 mT)^{3/2}]\exp(-(k_0/T) - (x^2/2R^2))\delta(x_0) \tag{6.19}$$

[79] In [217] the clumping in phase space due to Bose symmetry was also illustrated.

[80] The same interpretation of the multiplicity dependence of BEC was given in [211]. In this reference the nature of the "fake" coherence induced by fixing the multiplicity is even clearer, as one studies there explicitly coherence in a consistent quantum statistical formalism.

where

$$k_0/T = k^2/2\Delta^2 \; . \tag{6.20}$$

Here T is an effective temperature, R an effective radius, m the pion mass and Δ a constant with dimensions of momentum.

Let η_0 and η be the number densities before and after symmetrisation, respectively. In terms of $S(k,x)$ we have

$$\eta_0 = \int S(k,x)\, d^4k\, d^4x \tag{6.21}$$

and a corresponding expression for η with S replaced by the source function after symmetrisation. Then one finds [219] that η increases with η_0 and above a certain crtitical density η_0^{crit}, η diverges. This is interpreted in Ref. [219] as "passing".

The reader may be rightly puzzled by the fact that while η has a clear physical significance the number density η_0 and a fortiori its critical value have no physical significance, because in nature there does not exist a system of bosons the wave function of which is not symmetrised. Thus contrary to what is alluded to in Ref. [219], this paper does not address really the question how a condensate is reached. Indeed the physical factors which induce condensation are, for systems in (local) thermal and chemical equilibrium,[81] pressure and temperature and the symmetrisation is contained automatically in the distribution function

$$f = 1/[\exp[(E-\mu)/T] - 1] \tag{6.22}$$

in the term -1 in the denominator; E is the energy and μ the chemical potential.

To realise what is going on it is useful to observe that the increase of η_0 can be achieved by decreasing R and/or T. Thus η_0 can be substituted by one or both of these two physical quantities. Then the blow-up of the number density η can be thought of as occurring due to a decrease of T and/or R. However this is nothing but the well-known Bose–Einstein condensation phenomenon. While from a purely mathematical point of view the condensation effect can be achieved also by starting with a non-symmetrised wave function and symmetrising it afterwards "by hand", the causal, i.e. physical relationship is different: one starts with a bosonic, i.e. symmetrised system and obtains condensation by decreasing the temperature or by increasing the density of this *bosonic* system.

Another confusing interpretation in [219] relates to the observation made also in [217] that the symmetrisation produces a broadening of the multiplicity distribution (MD). In particular, starting with a Poisson MD for the non-symmetrised wf one ends up after symmetrisation with a negative binomial. This is a simple consequence of Bose statistics, and must not be associated with the so-called pasing effect. In fact for true lasers the opposite effect takes place. Before "condensing", i.e. below threshold their MD is in general broad and of negative binomial form corresponding to a chaotic (thermal) distribution while above threshold the laser condensate is produced and as such corresponds to a coherent state and therefore is characterised by a Poisson MD. Last but not least

[81] For lasers the determining dynamical factor is among other things the inversion.

the fact that this broadening increases with n is not, as suggested in [219,220], due to the approach to "lasing criticality", but simply to the fact that the larger the n, the larger the number of independent emitters is and the better the central limit theorem applies. This theorem states (see Section 2.2) that the field produced by a large number of independent sources is chaotic.

Finally a terminological remark appears necessary here. We believe that names like "paser" or pionic laser used in the papers quoted above are unjustified and misleading. The only characteristic which the systems considered in these papers possibly share with lasers is the condensate property, i.e. the bunching of particles in a given (momentum) state. However lasers are much more than just condensates; one of the main properties of lasers which distinguishes them from other condensates is the directionality, a problem which is not even mentioned in the "paser" literature.[82,83]

6.3. *The invariant Q variable in the space–time approach: higher-order correlations; "intermittency" in BEC?*

The issue of apparent power-like rapidity dependence of moments of MD was discussed in Section 6.1.1 where it was pointed out that this dependence could in principle be understood within the QS formalism without invoking the idea of intermittency. However this was not the end of the story because: (1) it was observed [192] that the power-like behaviour extends also to BEC data (in the invariant variable Q). This was surprising because up to that moment BEC data could usually be fitted by a Gaussian or exponential function, albeit these data did not extend to such small Q values as those measured in [192]; 2) Bialas [227] (see also [228,229]) proposed that the source itself has no fixed size, but is fluctuating from event to event with a power distribution of sizes. Since the measurements made in [192] were in the invariant variable Q rather than Δy, one had to understand whether the QS approach which implies fixed scales does not lead to a similar behaviour in Q.

Refs. [230,231] addressed this question and proved that, indeed, by starting from a space–time correlator with a fixed correlation length and a source distribution with a fixed radius, one gets after integrations over the unobserved variables a correlation function which is power like in a limited Q range.

In the following, we shall sketch how this happens. The two-particle Bose–Einstein correlation function is defined as

$$C_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \rho_2(\boldsymbol{k}_1,\boldsymbol{k}_2)/[\rho_1(\boldsymbol{k}_1)\cdot\rho_1(\boldsymbol{k}_2)] \tag{6.23}$$

where $\rho_1(\boldsymbol{k})$ and $\rho_2(\boldsymbol{k}_1,\boldsymbol{k}_2)$ are the one- and two-particle particle inclusive spectra, respectively.

The two-particle correlation function projected on Q^2 is

$$C_2(Q^2) = 1 + (\mathscr{I}_2(Q^2)/\mathscr{I}_{11}(Q^2)) \tag{6.24}$$

[82] For a model of directional coherence, not necessarily related to pion condensates, see [222]; experimental hints of this effect have possibly been seen in [223].

[83] For another investigation of the effect of symmetrization on the single inclusive cross section cf. [224,225]. In [225] second-order correlation functions are also considered (see also Ref. [226] for this topic).

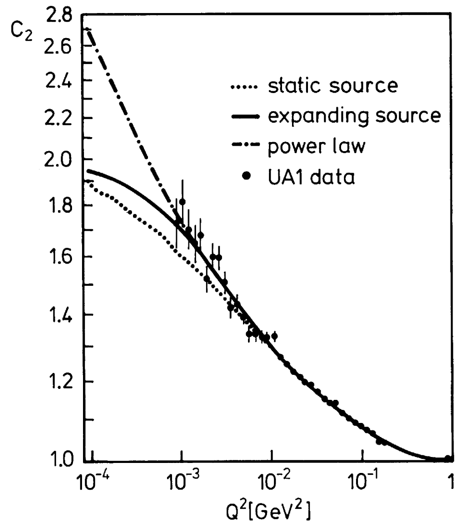

Fig. 21. $C_2(Q^2)$ for the static source model (dashed line) and for the expanding source model (solid line) compared to the UA1 data. The dotted line shows a power-law fit (from Ref. [230]).

with the integrals

$$\mathscr{I}_{11}(Q^2) = \int d\omega_1 \int d\omega_2 \delta[Q^2 + (k_1^\mu - k_2^\mu)^2]\rho_1(\boldsymbol{k}_1)\rho_1(\boldsymbol{k}_2) ,$$
$$\mathscr{I}_2(Q^2) = \int d\omega_1 \int d\omega_2 \, \delta[Q^2 + (k_1^\mu - k_2^\mu)^2](\rho_2(\boldsymbol{k}_1,\boldsymbol{k}_2) - \rho_1(\boldsymbol{k}_1)\rho_1(\boldsymbol{k}_2)) , \tag{6.25}$$

where $d\omega \equiv d^3k/(2\pi)^3 2E$ is the invariant phase space volume element. Two types of sources, a static one and an expanding one, were considered (cf. Section 4.8). The parameters employed are physically meaningful quantities in the sense that they give the lifetime, radii and correlation lengths of the source. This is not the case for the ad hoc parametrisation of $C_2(Q^2)$, e.g., with a Gaussian

$$C_2(Q^2) = 1 + \lambda_Q \, e^{-R_Q^2 Q^2} . \tag{6.26}$$

To illustrate the behaviour of the correlation function as a function of Q^2, one applied the formalism to describe UA1 data in the phase space region $|y| \leq 3.0$ and $k_\perp \geq 150$ MeV. In [230,231] $C_2(Q^2)$ was calculated for the static and for the expanding source by Monte Carlo integration (for the static source, approximate analytical results could be obtained only for $|y| > 1.5$, where they agree with the numerical results [231]). Fig. 21 shows the results of fits to the UA1 data[84] for the static source (dashed line) and for the expanding source (solid line), which were

[84] The $C_2(Q^2)$ data of [192] were normalised so that at large Q^2, $C_2(Q^2) \approx 1$. An explanation for the experimental observation of [192] that $C_2(Q^2)$ exceeds by a multiplicative factor of ~ 1.3 both the upper and lower "conventional" limits of 2 and 1, respectively, is discussed in Ref. [202].

obtained under the assumption of a purely chaotic source, $p_0 = 1$ (the data are consistent with an amount of $\leq 10\%$ coherence,[85] but the sensitivity to p_0 was not sufficient to further constrain the degree of coherence within these limits). For comparison, the result of a power law fit (dotted line) as suggested in Ref. [192]

$$C_2(Q^2) = a + b\cdot(Q^2/1\,\mathrm{GeV}^2)^{-\phi} \tag{6.27}$$

is also plotted.

One finds that the data can be well described both with the static and the expanding source model with reasonable values for the radii, lifetime and correlation length.

The results of above show that the power-like behaviour of C_2 (the same holds for higher order correlations) can be reproduced by assuming a conventional space–time source with fixed parameters, i.e., without invoking "intermittency".[86] This conclusion of [230] is strengthened by an explicit consideration of resonances in [231].

The advantages of the QS formalism as compared with the wave-function formalism emerge clearly also in the problem of higher-order correlations. In the QS formalism higher-order correlations are treated on the same footing as lower-order ones and emerge just as consequences of the form of the density matrix. Therefore questions found sometimes in the literature like "what is the influence of higher-order correlations on the lower one" do not even arise in QS and in fact do not make sense. We emphasised in Section 2.2.1 the importance of higher-order correlations for the phenomenological determination of the form of the correlation function and this applies in particular when a single variable like Q is used instead of the six independent degrees of freedom inherent in the correlator. For this reason the space–time integration at fixed Q has been used recently [35] also in the study of higher-order correlations and applied to the NA22 data [233]. It was found among other things that an expanding source with fixed parameters as defined in Section 4 can account for the data up to and including the fourth order, confirming in a first approximation the Gaussian form of the density matrix. The fact that previous attempts in this direction like [233,34] met with difficulties may be due to the fact that in these two experimental studies the QO formalism in momentum space was used and possibly also because no simultaneous fit of all orders of correlations was performed, as was the case in [35].

7. Critical discussion and outlook

For historical reasons related to the fact that the GGLP effect was observed for the first time in annihilation at rest when (almost) exclusive reactions were studied, the theory of BEC was initially based on the wave-function formalism. This formalism is not appropriate for inclusive reactions in high-energy physics, among other things because it yields a correlation function which depends

[85] The older UA-1 data [31] were limited to larger Q^2 values. Furthermore, the quantum statistical interpretation [32] of these data is different from that in [230], which is based on space–time concepts. This explains the different values of chaoticity obtained in [230,231], on the one hand and in Refs. [31,32] on the other hand.

[86] A similar point of view is expressed in [232] for the particular case of bremsstrahlung photons.

only on the momentum difference q and not also on the sum of momenta, it does not take into account isospin, and cannot treat adequately coherence. This last property is essential as it leads to one of the most important applications of BEC, i.e. to condensates. Furthermore, this property strongly affects another essential application of BEC, namely the determination of sizes, lifetimes, correlation lengths and correlation times of sources. The ad hoc parametrisation of the correlation function under the form (2.8) where λ is supposed to take into account (in)coherence is unsatisfactory. An improvement based on an analogy with quantum optics leads to an additional term (with the same number of parameters). A more complete and more correct treatment of BEC is provided by the space–time formalism of classical currents.

Almost all studies of BEC assume a Gaussian density matrix. Possible deviations from this form could and should be looked for by studying higher-order correlations.

The classical current approach is based on an exact solution of the equations of quantum field theory and it can be considered at present the most advanced and complete description of BEC. It introduces a primordial correlator of currents which is characterised by finite correlation lengths and correlation times. The geometry of the source is an independent property of the system and it is here that the traditional radii and lifetimes enter. The form of the geometry and of the correlator is not given by the theory and it is up to experiment to determine these.

For a Gaussian density matrix the space–time approach within the classical current formalism leads to a minimum of 10 independent parameters; these include geometrical and dynamical scales as well as the chaoticity. Phenomenologically, these scales can be separated only by considering simultaneously single and double inclusive cross sections. Experimentally, this separation as well as the determination of all parameters has not yet been performed and constitutes an important task for the future. *This should be done not only for particle reactions but also for heavy ion reactions, as an alternative to the quasi-hydrodynamical approach which cannot provide this separation.* It would also be desirable to extend the parametrisation for an expanding source by dropping the assumption of boost invariance.

Besides the conventional $++$ or $--$ pion correlations there exist also $+-$ correlations, which become important for sources of small lifetimes. These "surprising" field theoretical effects represent squeezed states, which unlike what happens in optics, appear in particle physics "for free". These effects can be used to investigate the difference between classical and quantum currents. Their detection constitutes one of the most important challenges for future experiments.

BEC are influenced by final state interactions. Coulomb final state interactions do not play a major part except at very small q. These small values have apparently not yet been reached in experiment and it is questionable whether they will be reached in the foreseeable future. In heavy ion reactions this is due to the large values of radii which multiply q and even for typical scales of 1 fm the present resolution of detectors is not sufficient to make this effect very important. Nevertheless, since Coulomb corrections have been studied so far only within the wave-function formalism and usually by applying the Schrödinger equation, it would be desirable to extend this study by considering the Klein–Gordon equation which is more appropriate for mesons. Even more interesting would be to study Coulomb corrections within the classical current formalism.

Resonances play an important part in final state interactions and progress has been achieved in their understanding, in particular in heavy ion reactions, where their influence has been investigated using solutions of the equations of hydrodynamics within the Wigner function formalism.

BEC have been investigated in e^+–e^-, hadron–hadron and heavy ion reactions; however a systematic comparison of results using the same parametrisation and normalisation is yet to be done.[87]

Correlations are intimately related to multiplicity distributions which can serve as complementary tools in the determination of the parameters of sources. Therefore a systematic investigation of BEC and multiplicity distributions in the same phase space region is desirable. How useful this can be has been shown by proving in this way the influence of long-range correlations on BEC.

BEC can be useful in heavy ion reactions and for the search of quark–gluon plasma if in particular one of the two conditions is satisfied:

(A) the investigation is based on full-fledged hydrodynamics, implying the solution of the equations of hydrodynamics with explicit consideration of the equation of state.

(B) the investigation is based on the classical current space–time approach.

Case (A) has the advantage that the dependence of the equation of state on the phase transition may be reflected also in single inclusive cross sections and in the BEC. (So far, however, the sensitivity of BEC on the equation of state has not yet been proven with present data [234].) It has however the disadvantage that its applicability is restricted because it is based on the Wigner function, which is a particular case of the classical current formalism, and which is useful only for small q and not too strong correlations between momenta k and coordinates x.

Case (B) has the disadvantage that there is no contact with the equation of state. However it has the advantage it is not restricted to small q and weak correlations between k and x.

Unfortunately, many of the theoretical papers on BEC in heavy ion reactions do not satisfy either condition (A) or condition (B). This is the case with most of the quasi-hydrodynamical papers which use a parametrisation of the source function based on qualitative hydrodynamical considerations without the use of an equation of state and without solving the equations of hydrodynamics. This quasi-hydrodynamical approach has the disadvantages of (A) and (B) but none of their advantages.[88]

There are many yet unsolved problems in the investigation of BEC, some of them of theoretical, but most of them of experimental nature (see also Ref. [235]).

While an analysis of the experimental BEC deserves a special review, some of the obvious reasons for this unsatisfactory experimental situation are:

(1) Most of the BEC experiments performed so far use inadequate detectors, because they are not dedicated experiments but rather by-products of experiments planned for other purposes. What is needed among other things is track-by-track detection and improved identification of particles.

(2) Insufficient statistics. An improvement of statistics especially at small q by at least one order of magnitude is necessary to address some of the problems enumerated above.

[87] At present even for the same type of reaction different normalisations are frequently used and this can lead to *apparently* different results, as exemplified in the case of NA22 and UA1 data. Tests for a better understanding and elimination of these discrepancies have been proposed (see Section 6.1.3).

[88] Quasi-hydrodynamics as compared with hydrodynamics has the supplementary disadvantage that it does not allow a separation between geometrical radii and correlation lengths. Moreover it does not have even the excuse of simplicity, since the number of free parameters in the quasi-hydrodynamical approach is as large as that in the classical current space–time approach.

(3) Incorrect or incomplete parametrisations of the correlation functions. Very often and in part because of (2) not all six independent variables of C_2 are measured, but projection of these. Very popular among these projections is the relativistically invariant variable Q. This is not a good variable for BEC studies, because among other things it mixes the space and time variables in an uncontrollable way. Furthermore, in most parametrisations, coherence is not (or inadequately) considered.

(4) Inadequate normalisations. Practically, all BEC experiments use a normalisation procedure of the correlation function which does not correspond to its definition. This definition relates the double (or higher order) inclusive cross section to the product of single inclusive cross sections and not to an "uncorrelated background".

The solution of the problems mentioned above will make of boson interferometry what it is supposed to be: a reliable method for the determination of sizes, lifetimes, correlation lengths, and coherence of sources in subatomic physics.

A more pedagogical presentation of the theory of Bose–Einstein correlations, which discusses also its quantum optical context, including a comparison between the HBT and the GGLP effects and between photon and hadron intensity interferometry can be found in the book by the author [5]. Some of the most representative theoretical and experimental papers on BEC and which are frequently quoted within the present review have been reprinted in a single volume in Ref. [236].

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